Dynamics and Relativity: Example Sheet 4

Professor David Tong, February 2013

1. A clock $\mathcal{C}$ is at rest at the spatial origin of an inertial frame S. A second clock $\mathcal{C}'$ is at rest at the spatial origin of an inertial frame S' moving with constant speed u relative to S. The clocks read $t = t' = 0$ when the two spatial origins coincide.

When $\mathcal{C}'$ reads t'_2 it receives a radio signal from $\mathcal{C}$ sent when $\mathcal{C}$ reads t_1. Draw a space-time diagram showing this process.

Determine the space-time coordinates (ct_2, x_2) in S of the point (event) at which $\mathcal{C}'$ receives the radio signal. Hence show that

$$t_1 = t'_2 \sqrt{\frac{1 - u/c}{1 + u/c}}$$

2. S and S' are inertial frames in a 2-dimensional space-time, the origins of which coincide. Observers O and O' are at the spatial origins of S and S' respectively. Observer O' moves at velocity u relative to observer O, where $u > 0$. Observer O observes a particle P passing through the origin and moving with velocity v, where $v < u$. The observer O' observes P moving with velocity $-v$.

Draw a space-time diagram, from O's point of view, to illustrate this situation. Use the relativistic velocity transformation law to show that

$$v = \frac{c^2}{u}\left(1 - \sqrt{1 - \frac{u^2}{c^2}}\right)$$

Writing $u/c = \tanh\varphi$, find an expression for v/c in terms of the rapidity φ.

3. Write down the Lorentz transformation law for the energy and relativistic 3-momentum of a particle.

In an inertial frame S, a photon with energy E moves in the $x-y$ plane at an angle θ relative to the x-axis. Show that in a second frame S' whose relative speed is u directed in the x-direction, the energy and angle are given by

$$E' = \gamma E(1 - \beta\cos\theta) \quad \text{and} \quad \cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

where $\beta = u/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Write down E and $\cos\theta$ as functions of E' and $\cos\theta'$.

For a photon moving in the x-direction, derive the *relativistic Doppler effect* by showing that there is a frequency change by a factor

$$\sqrt{\frac{1-\beta}{1+\beta}}$$

(*) How does this relate to question 1?

A source of photons is at rest in S'. Derive the *headlight effect* by showing that as $u \to c$, the photons emitted in the forward direction ($\cos\theta' > 0$) in S are concentrated in a narrow cone about $\theta = 0$.

(*) Show that the semi-angle of the cone is approximately $\sqrt{2(1-\beta)}$.

4. Pulsars are stars which emit pulses of radiation at regular intervals. Bob and Ann count pulses from a very distant pulsar in the y direction. Ann travels at a speed given by $\beta \equiv u/c = 24/25$ in the x-direction for seven years and then comes back at the same speed, while Bob stays at home. At the end of the trip they have counted the same number of pulses. Use question 3 (with $\theta = \frac{1}{2}\pi$) to show that on return she has aged by 14 years and he by 50.

Obtain the corresponding result when the pulsar is in the x direction (question 3 with $\theta = 0$), drawing a space-time diagram to show why Bob and Ann count the same number of pulses.

5. A particle with 4-momentum P is detected by an observer whose four-velocity is U. Working in the rest frame of the observer, express $P \cdot U$ in terms of the mass of the particle and its speed v in the observer's rest frame. Show that

$$\frac{v}{c} = \sqrt{1 - \frac{(P \cdot P)c^2}{(P \cdot U)^2}}$$

6. A particle of rest mass m_0 disintegrates into two particles of rest masses m_1 and m_2. Use conservation of relativistic energy and relativistic 3-momentum to show that the energies E_1 and E_2 of the particles in the rest frame of the original particle are given by

$$E_1 = c^2 \, \frac{m_0^2 + m_1^2 - m_2^2}{2m_0} \qquad \text{and} \qquad E_2 = c^2 \, \frac{m_0^2 + m_2^2 - m_1^2}{2m_0}$$

Let P_0 be the 4-momentum of the original particle, and let and P_1 and P_2 be the 4-momenta of the product particles. Derive the above results by expanding the invariant quantities $(P_0 - P_1) \cdot (P_0 - P_1)$ and $(P_0 - P_2) \cdot (P_0 - P_2)$.

7. A photon (of zero rest mass) collides with an electron of rest mass m which is initially at rest in the laboratory frame. Show that the angle θ by which the photon is deflected (measured in the laboratory frame) is related to the magnitudes p and q of its initial and final momenta by

$$2 \sin^2 \frac{1}{2}\theta = \frac{mc}{q} - \frac{mc}{p}.$$

8. In a laboratory frame a particle of rest mass m_1 has energy E_1, and a second particle of rest mass m_2 is at rest. By considering the scalar quantity $(P_1 + P_2) \cdot (P_1 + P_2)$, or otherwise, show the combined energy in the centre of momentum frame (i.e. the frame in which the total 3-momentum is zero) is

$$\sqrt{m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2}.$$

Hence show that in a collision between a proton with energy E and a proton at rest it is not possible to create a proton-antiproton pair (in addition to the original protons) if $E < 7mc^2$, where m is the rest mass of the proton and also of the antiproton.

9. A rocket ejects exhaust at constant speed u relative to itself by a process that conserves relativistic energy and momentum (but not mass!). Let m be the rest mass of the rocket when the rocket has speed v measured in the *initial* rest frame of the rocket.

By equating the 4-momentum of the rocket P with the total 4-momentum at a later time when the 4-momentum of the rocket is $P + dP$, show that

$$\frac{d(m\gamma v)}{dv} = \left(\frac{v-u}{1-uv/c^2}\right)\frac{d(m\gamma)}{dv},$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. Rearrange this as a differential equation for $m\gamma$ and derive the relativistic rocket equation

$$v = c\tanh\left(\frac{u}{c}\log r\right)$$

where $r = m_0/m$ and m_0 is the initial rest mass of the rocket.

Now suppose that the rocket has an ideal photon drive, so that all matter to be ejected is converted first to photons by means of positron-electron annihilation. By considering directly initial and final energy and momentum (taking into account the total energy and momentum of the ejected photons, which you do not need to calculate), derive the relativistic rocket equation for the case of a photon-drive rocket.