

The Standard Model: Example Sheet 1

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1. Show that $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$ satisfies the Lorentz algebra,

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i(\eta^{\nu\rho} \sigma^{\mu\sigma} - \eta^{\nu\sigma} \sigma^{\mu\rho} + \eta^{\mu\sigma} \sigma^{\nu\rho} - \eta^{\mu\rho} \sigma^{\nu\sigma})$$

Hint: you may find it useful to first rewrite: $\sigma^{\mu\nu} = \frac{i}{2}(\eta^{\mu\nu} - \sigma^\nu \bar{\sigma}^\mu)$. At some stage of the calculation, you may also find it useful to prove the result $(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = 2\eta^{\mu\nu} \mathbf{1}_2$.

2. Under a Lorentz transformation, a left-handed Weyl spinor ψ_L and right-handed spinor ψ_R transform as

$$(\psi_L)_\alpha \rightarrow S_\alpha^\beta (\psi_L)_\beta \quad \text{and} \quad (\psi_R)_{\dot{\alpha}} \rightarrow (S^*)_{\dot{\alpha}}^{\dot{\beta}} (\psi_R)_{\dot{\beta}} \quad (1)$$

with $S \in SL(2, \mathbb{C})$. (Here the dotted index $\dot{\alpha} = 1, 2$ is used to reflect the fact that these spinors transform in different representations. What we call $(\psi_R)_{\dot{\alpha}}$ here is called $\bar{\psi}_{\dot{\alpha}}$ in the Supersymmetry course.) Show that:

- i) $(S^{-1})_\beta^\alpha = \epsilon^{\alpha\gamma} S_\gamma^\lambda \epsilon_{\lambda\beta}$
- ii) $(\psi_L)^\alpha = \epsilon^{\alpha\beta} (\psi_L)_\beta$ transforms as $(\psi_L)^\alpha \rightarrow (\psi_L)^\beta (S^{-1})_\beta^\alpha$
- iii) $\psi_L \chi_L = (\psi_L)^\alpha (\chi_L)_\alpha$ is a Lorentz scalar.
- iv) $(\psi_R)^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} (\psi_R)_{\dot{\beta}}$ transforms as $(\psi_R)^{\dot{\alpha}} \rightarrow (S^{-1\dagger})^{\dot{\alpha}}_{\dot{\beta}} (\psi_R)^{\dot{\beta}}$.
- v) $\bar{\psi}_R \psi_L = (\psi_R^*)^\alpha (\psi_L)_\alpha$ is a Lorentz scalar.

3. Define $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$ and $\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$. Show that $\bar{\sigma}^{\mu\nu} = (\sigma^{\mu\nu})^\dagger$. Let

$$S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$$

with $\omega_{\mu\nu}$ a collection of numbers that specify the Lorentz transformation. Show

$$S^{-1\dagger} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right).$$

(It suffices to show this for infinitesimal $\omega_{\mu\nu}$ and then use general properties of Lie groups.) As an aside: combined with the result from Questions 2iv), this shows that a Dirac spinor should be viewed as having indices $\psi^T = ((\psi_L)_\alpha, (\psi_R)^{\dot{\alpha}})$.

4. A pair of Weyl spinors ψ_L and ψ_R have both a Dirac mass $M \in \mathbb{R}$ and Majorana masses $m_1, m_2 \in \mathbb{C}$. These appear in the Lagrangian as

$$\mathcal{L}_{\text{mass}} = -M(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) + \frac{m_1}{2}\psi_L\psi_L + \frac{m_1^*}{2}\bar{\psi}_L\bar{\psi}_L + \frac{m_2}{2}\psi_R\psi_R + \frac{m_2^*}{2}\bar{\psi}_R\bar{\psi}_R .$$

What are the physical masses of fermionic particles in this theory?

5*. The Lie algebra-valued gauge potential $A_\mu = A_\mu^A T^A$ transforms under a gauge symmetry G as

$$A_\mu \rightarrow \Omega A_\mu \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1}$$

where g is the coupling and $\Omega(x) = e^{ig\alpha(x)} \in G$ with $\alpha(x) = \alpha^A(x) T^A$. Show that:

- i) the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ transforms as $F_{\mu\nu} \rightarrow \Omega F_{\mu\nu} \Omega^{-1}$.
- ii) under an infinitesimal gauge transformation $\delta A_\mu = \mathcal{D}_\mu \alpha$ where the covariant derivative is defined by $\mathcal{D}_\mu \alpha = \partial_\mu \alpha - ig[A_\mu, \alpha]$.
- iii) under an infinitesimal gauge transformation $\delta F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \alpha = ig[\alpha, F_{\mu\nu}]$.
- iv) the field strength obeys the Bianchi identity $\mathcal{D}_\mu^* F^{\mu\nu} = 0$.
- v) the action

$$S = -\frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu}$$

is gauge invariant.

- vi) the equation of motion that follows by varying the action with respect to the fields A_μ^A is $\mathcal{D}_\mu F^{\mu\nu} = 0$.

A scalar ϕ in the fundamental \mathbf{N} representation of $SU(N)$ transforms as $\phi \rightarrow \Omega \phi$. How does the covariant derivative $\mathcal{D}_\mu \phi = \partial_\mu \phi - ig A_\mu \phi$ transform?

6*. The chiral basis of gamma matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. For a Dirac spinor, $\bar{\psi} = \psi^\dagger \gamma^0$. The equation of motion for a $U(1)$ gauge field coupled to a Dirac fermion is

$$\partial_\nu F^{\mu\nu} = e \bar{\psi} \gamma^\mu \psi .$$

Use the transformation properties of a Dirac spinor under C, P, and T, to derive the corresponding transformation properties of A_μ that ensure this equation remains invariant.

The *theta term* is an extra term that can be added to the Maxwell (or Yang-Mills) action. For Maxwell theory, it takes the form

$$S_\theta = \int d^4x \, F_{\mu\nu} {}^\star F^{\mu\nu} .$$

How does this transform under C, P, and T?

7. Show that $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$ are both real. Consider the mass term

$$\mathcal{L}_{\text{mass}} = m_1\bar{\psi}\psi + im_2\bar{\psi}\gamma^5\psi .$$

Using the chiral basis of gamma matrices, write this mass term in terms of Weyl spinors, with $\psi^T = (\psi_L, \psi_R)$. Find a transformation of ψ_L and ψ_R such that this theory is invariant under parity. How does parity act on the Dirac spinor?

8. In $d = 2 + 1$ dimensions, with signature $(+, -, -)$, we can take the basis of purely imaginary 2×2 gamma matrices $\gamma^\mu = (\sigma^2, i\sigma^1, i\sigma^3)$. A massive Dirac fermion has action

$$S = - \int d^d x \left(i\bar{\psi}\gamma^\mu \partial_\mu \psi - M\bar{\psi}\psi \right) .$$

Parity is defined as $P : x^1 \mapsto -x^1$, with x^0 and x^2 untouched. Why is this the right definition, rather than the more usual $\mathbf{x} \rightarrow -\mathbf{x}$?

Find an action of parity, charge conjugation, and time reversal for a massless fermion that leaves the action invariant. Which of these symmetries are broken by the mass term?