The Standard Model: Example Sheet 2

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1. Construct a theory of a complex scalar field ϕ with a polynomial potential that has a spontaneously broken $\mathbb{Z}_N$ symmetry and identify its ground states.

2. Let $M(x)$ be an $N \times N$ complex matrix field with action

$$S = \int d^4x \; \mathrm{Tr} \left(\partial^\mu M^\dagger \partial_\mu M - k\, M^\dagger M - \frac{\lambda}{2}\, M^\dagger M M^\dagger M \right)$$

with $\lambda > 0$.

i) Show that this theory is invariant under the transformations $M \to AMB^\dagger$ with $A, B \in U(N)$. Show that there is a subgroup $U(1) \subset U(N) \times U(N)$ that doesn't act on M.

ii) Show that the symmetry is spontaneously broken if $k < 0$, with the ground state obeying $M_0^\dagger M_0 = v^2 \mathbb{1}$ for some v^2. What is the unbroken symmetry group? Write $\mathcal{M}_0$ as a group coset and determine the number of Goldstone bosons.

iii) Consider the deformed action

$$S' = S + \int d^4x \; h \left(\det M + \det M^\dagger \right) .$$

What is the symmetry group of this action? Assuming that the ground state still sits at $M_0^\dagger M_0 = v^2 \mathbb{1}$ for some $v^2 \neq 0$, how many Goldstone bosons are there?

3*. An $SU(2)$ gauge theory coupled is coupled to a scalar ϕ in the fundamental representation. We write ϕ^a with $a = 1, 2$. The action is

$$S = \int d^4x \; \left(-\frac{1}{2} \mathrm{Tr}\,(F_{\mu\nu} F^{\mu\nu}) + \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - \frac{\lambda}{2} (\phi^\dagger \phi - v^2)^2 \right).$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and $\mathcal{D}_\mu \phi = \partial_\mu \phi - igA_\mu \phi$. [The $SU(2)$ generators in the fundamental representation are $\vec{T} = \frac{1}{2}\vec{\sigma}$, with $\vec{\sigma}$ the usual triplet of Pauli matrices.]

What are the masses of the particles in this theory?

Suppose now that we have $SU(N)$ gauge theory coupled to a single scalar ϕ in the fundamental representation. If the scalar condenses, what is the surviving symmetry? How many gauge bosons get a mass? How many components of ϕ must be eaten to achieve this?

4. A real scalar field ϕ in the adjoint representation can be viewed as taking values in the Lie algebra: $\phi = \phi^A T^A$, with T^A the generators. For gauge group $G = SU(N)$, this means that ϕ is a traceless $N \times N$ matrix with covariant derivative

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - ig[A_\mu, \phi]$$

Suppose that the potential is minimised by $\phi = \phi_0$. Explain why we can always take ϕ_0 to be diagonal,

$$\phi_0 = \mathrm{diag}(v_1, \ldots, v_N)$$

with $\sum_a v_a = 0$ and $v_a \leq v_{a+1}$. Describe how the symmetry breaking pattern depends on the eigenvalues v_a.

5*. The coupling constant g for an $SU(N_c)$ gauge theory, coupled to N_f massless Dirac fermions, runs at one loop as

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{1}{3(4\pi)^2}(11N_c - 2N_f)\log\frac{\Lambda_{UV}^2}{\mu^2}$$

For QCD, the coupling constant $\alpha_s = g_s^2/4\pi$ takes value $\alpha_s(\mu) \approx 0.12$ at $\mu = M_Z \approx 90$ GeV. Determine the value of Λ_{QCD} assuming that all quarks lighter than M_Z are actually massless. Can you get a more realistic approximation to Λ_{QCD}?

6. Let $T(R)$ be the generator of a Lie algebra g in the representation R. The quadratic Casimir $C(R)$ and Dynkin index $I(R)$ are defined as

$$T^A T^A = C(R)\,\mathbb{1} \quad \text{and} \quad \mathrm{Tr}\, T^A T^B = \frac{1}{2} I(R)\delta^{AB}$$

Show that $2C(R)\dim(R) = I(R)\dim(G)$. Hence determine $C(R)$ for the fundamental and anti-fundamental representations of $SU(N)$. Calculate $C(\mathrm{adj})$ for the adjoint representation of $SU(2)$.

7. The chiral Lagrangian is

$$\mathcal{L}_{\mathrm{pion}} = \frac{f_\pi^2}{4}\mathrm{tr}(\partial^\mu U^\dagger\, \partial_\mu U)$$

with $U(x) = e^{2i\pi(x)/f_\pi}$ where $\pi(x)$ is valued in $su(N_f)$. Show that the quadratic and quartic terms in π are

$$\mathcal{L}_{\mathrm{pion}} = \mathrm{tr}\,(\partial_\mu \pi)^2 - \frac{2}{3f_\pi^2}\mathrm{tr}\,\left(\pi^2(\partial_\mu\pi)^2 - (\pi\partial_\mu\pi)^2\right) + \ldots$$

For $N_f = 2$, with generators $T^a = \frac{1}{2}\sigma^a$, show that the quartic terms take the form

$$\mathcal{L}_{\mathrm{int}} = -\frac{1}{6f_\pi^2}\left(\pi^a\pi^a\partial\pi^b\partial\pi^b - \pi^a\partial\pi^a\pi^b\partial\pi^b\right)$$

8. For $N_f = 3$, the Goldstone bosons are pions, kaons and the eta. They sit inside the matrix π as

$$\pi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

These mesons obtain masses from the term

$$\mathcal{L}_{\text{mass}} \sim f_\pi \text{tr}\Big((M + M^\dagger)\pi^2\Big)$$

where $M = \text{diag}(m_u, m_d, m_s)$ is the matrix of (renormalised) quark masses. Show that

$$\frac{m_{K^+}^2 - m_{K^0}^2}{m_\pi^2} = \frac{m_u - m_d}{m_u + m_d}$$

If we approximate $m_u \approx m_d$, derive the *Gell-Mann-Okubo* relation

$$4m_K^2 \approx 3m_\eta^2 + m_\pi^2$$

Compare this prediction against the measured masses of particles.

9. Consider the $SU(2)$ gauge configuration

$$A_\mu = \frac{1}{g}\frac{1}{x^2 + \rho^2}\,\eta^a_{\mu\nu} x^\nu \sigma^a$$

with ρ parameter and $\eta^a_{\mu\nu}$ a collection of three 4×4 *'t Hooft matrices* given by

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \;,\; \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \;,\; \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Show that the corresponding field strength is given by

$$F_{\mu\nu} = -\frac{1}{g}\frac{2\rho^2}{(x^2 + \rho^2)^2}\eta^a_{\mu\nu}\sigma^a$$

Why does this solve the Euclidean Yang-Mills equation of motion $\mathcal{D}_\mu F^{\mu\nu} = 0$? Compute the action

$$S = \frac{1}{2}\int d^4x \; \text{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

Note: You will need the identity

$$\epsilon^{abc}\eta^a_{\mu\rho}\eta^b_{\nu\sigma} = \eta^c_{\mu\nu}\delta_{\rho\sigma} + \eta^c_{\rho\sigma}\delta_{\mu\nu} - \eta^c_{\mu\sigma}\delta_{\rho\nu} - \eta^c_{\rho\nu}\delta_{\mu\sigma}$$