4. Introducing Conformal Field Theory

The purpose of this section is to get comfortable with the basic language of two dimensional conformal field theory[4]. This is a topic which has many applications outside of string theory, most notably in statistical physics where it offers a description of critical phenomena. Moreover, it turns out that conformal field theories in two dimensions provide rare examples of interacting, yet exactly solvable, quantum field theories. In recent years, attention has focussed on conformal field theories in higher dimensions due to their role in the AdS/CFT correspondence.

A *conformal transformation* is a change of coordinates $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ such that the metric changes by

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma) \tag{4.1}$$

A *conformal field theory* (CFT) is a field theory which is invariant under these transformations. This means that the physics of the theory looks the same at all length scales. Conformal field theories care about angles, but not about distances.

A transformation of the form (4.1) has a different interpretation depending on whether we are considering a fixed background metric $g_{\alpha\beta}$, or a dynamical background metric. When the metric is dynamical, the transformation is a diffeomorphism; this is a gauge symmetry. When the background is fixed, the transformation should be thought of as an honest, physical symmetry, taking the point σ^α to point $\tilde{\sigma}^\alpha$. This is now a global symmetry with the corresponding conserved currents.

In the context of string theory in the Polyakov formalism, the metric is dynamical and the transformations (4.1) are residual gauge transformations: diffeomorphisms which can be undone by a Weyl transformation.

In contrast, in this section we will be primarily interested in theories defined on fixed backgrounds. Apart from a few noticeable exceptions, we will usually take this background to be flat. This is the situation that we are used to when studying quantum field theory.

[4]Much of the material covered in this section was first described in the ground breaking paper by Belavin, Polyakov and Zamalodchikov, "*Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*", Nucl. Phys. B241 (1984). The application to string theory was explained by Friedan, Martinec and Shenker in "*Conformal Invariance, Supersymmetry and String Theory*", Nucl. Phys. B271 (1986). The canonical reference for learning conformal field theory is the excellent review by Ginsparg. A link can be found on the course webpage.

Of course, we can alternate between thinking of theories as defined on fixed or fluctuating backgrounds. Any theory of 2d gravity which enjoys both diffeomorphism and Weyl invariance will reduce to a conformally invariant theory when the background metric is fixed. Similarly, any conformally invariant theory can be coupled to 2d gravity where it will give rise to a classical theory which enjoys both diffeomorphism and Weyl invariance. Notice the caveat "classical"! In some sense, the whole point of this course is to understand when this last statement also holds at the quantum level.

Even though conformal field theories are a subset of quantum field theories, the language used to describe them is a little different. This is partly out of necessity. Invariance under the transformation (4.1) can only hold if the theory has no preferred length scale. But this means that there can be nothing in the theory like a mass or a Compton wavelength. In other words, conformal field theories only support massless excitations. The questions that we ask are not those of particles and S-matrices. Instead we will be concerned with correlation functions and the behaviour of different operators under conformal transformations.

4.0.1 Euclidean Space

Although we're ultimately interested in Minkowski signature worldsheets, it will be much simpler and elegant if we work instead with Euclidean worldsheets. There's no funny business here — everything we do could also be formulated in Minkowski space.

The Euclidean worldsheet coordinates are $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$ and it will prove useful to form the complex coordinates,

$$z = \sigma^1 + i\sigma^2 \quad \text{and} \quad \bar{z} = \sigma^1 - i\sigma^2$$

which are the Euclidean analogue of the lightcone coordinates. Motivated by this analogy, it is common to refer to holomorphic functions as "left-moving" and anti-holomorphic functions as "right-moving".

The holomorphic derivatives are

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$

These obey $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. We will usually work in flat Euclidean space, with metric

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz\, d\bar{z} \tag{4.2}$$

In components, this flat metric reads

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad \text{and} \quad g_{z\bar{z}} = \frac{1}{2}$$

With this convention, the measure factor is $dzd\bar{z} = 2d\sigma^1 d\sigma^2$. We define the delta-function such that $\int d^2z\, \delta(z,\bar{z}) = 1$. Notice that because we also have $\int d^2\sigma\, \delta(\sigma) = 1$, this means that there is a factor of 2 difference between the two delta functions. Vectors naturally have their indices up: $v^z = (v^1 + iv^2)$ and $v^{\bar{z}} = (v^1 - iv^2)$. When indices are down, the vectors are $v_z = \frac{1}{2}(v^1 - iv^2)$ and $v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2)$.

4.0.2 The Holomorphy of Conformal Transformations

In the complex Euclidean coordinates z and $\bar{z}$, conformal transformations of flat space are simple: they are any holomorphic change of coordinates,

$$z \to z' = f(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z})$$

Under this transformation, $ds^2 = dzd\bar{z} \;\to\; |df/dz|^2\, dzd\bar{z}$, which indeed takes the form (4.1). Note that we have an infinite number of conformal transformations — in fact, a whole functions worth $f(z)$. This is special to conformal field theories in two dimensions. In higher dimensions, the space of conformal transformations is a finite dimensional group. For theories defined on $\mathbf{R}^{p,q}$, the conformal group is $SO(p+1, q+1)$ when $p + q > 2$.

A couple of particularly simple and important examples of 2d conformal transformations are

- $z \to z + a$: This is a translation.
- $z \to \zeta z$: This is a rotation for $|\zeta| = 1$ and a scale transformation (also known as a *dilatation*) for real $\zeta \neq 1$.

For many purposes, it's simplest to treat z and $\bar{z}$ as independent variables. In doing this, we're really extending the worldsheet from $\mathbf{R}^2$ to $\mathbf{C}^2$. This will allow us to make use of various theorems from complex methods. However, at the end of the day we should remember that we're really sitting on the real slice $\mathbf{R}^2 \subset \mathbf{C}^2$ defined by $\bar{z} = z^\star$.

4.1 Classical Aspects

We start by deriving some properties of classical theories which are invariant under conformal transformations (4.1).

4.1.1 The Stress-Energy Tensor

One of the most important objects in any field theory is the *stress-energy tensor* (also known as the energy-momentum tensor). This is defined in the usual way as the matrix of conserved currents which arise from translational invariance,

$$\delta\sigma^\alpha = \epsilon^\alpha \ .$$

In flat spacetime, a translation is a special case of a conformal transformation.

There's a cute way to derive the stress-energy tensor in any theory. Suppose for the moment that we are in flat space $g_{\alpha\beta} = \eta_{\alpha\beta}$. Recall that we can usually derive conserved currents by promoting the constant parameter ϵ that appears in the symmetry to a function of the spacetime coordinates. The change in the action must then be of the form,

$$\delta S = \int d^2\sigma \ J^\alpha \, \partial_\alpha \epsilon \tag{4.3}$$

for some function of the fields, J^α. This ensures that the variation of the action vanishes when ϵ is constant, which is of course the definition of a symmetry. But when the equations of motion are satisfied, we must have $\delta S = 0$ for all variations $\epsilon(\sigma)$, not just constant ϵ. This means that when the equations of motion are obeyed, J^α must satisfy

$$\partial_\alpha J^\alpha = 0$$

The function J^α is our conserved current.

Let's see how this works for translational invariance. If we promote ϵ to a function of the worldsheet variables, the change of the action must be of the form (4.3). But what is J^α? At this point we do the cute thing. Consider the same theory, but now coupled to a dynamical background metric $g_{\alpha\beta}(\sigma)$. In other words, coupled to gravity. Then we could view the transformation

$$\delta\sigma^\alpha = \epsilon^\alpha(\sigma)$$

as a diffeomorphism and we know that the theory is invariant as long as we make the corresponding change to the metric

$$\delta g_{\alpha\beta} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha \ .$$

This means that if we just make the transformation of the coordinates in our original theory, then the change in the action must be the opposite of what we get if we just

transform the metric. (Because doing both together leaves the action invariant). So we have

$$\delta S = -\int d^2\sigma \, \frac{\partial S}{\partial g_{\alpha\beta}} \, \delta g_{\alpha\beta} = -2 \int d^2\sigma \, \frac{\partial S}{\partial g_{\alpha\beta}} \, \partial_\alpha \epsilon_\beta$$

Note that $\partial S/\partial g_{\alpha\beta}$ in this expression is really a functional derivatives but we won't be careful about using notation to indicate this. We now have the conserved current arising from translational invariance. We will add a normalization constant which is standard in string theory (although not necessarily in other areas) and define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}} \tag{4.4}$$

If we have a flat worldsheet, we evaluate $T_{\alpha\beta}$ on $g_{\alpha\beta} = \delta_{\alpha\beta}$ and the resulting expression obeys $\partial^\alpha T_{\alpha\beta} = 0$. If we're working on a curved worldsheet, then the energy-momentum tensor is covariantly conserved, $\nabla^\alpha T_{\alpha\beta} = 0$.

The Stress-Energy Tensor is Traceless

In conformal theories, $T_{\alpha\beta}$ has a very important property: its trace vanishes. To see this, let's vary the action with respect to a scale transformation which is a special case of a conformal transformation,

$$\delta g_{\alpha\beta} = \epsilon g_{\alpha\beta} \tag{4.5}$$

Then we have

$$\delta S = \int d^2\sigma \, \frac{\partial S}{\partial g_{\alpha\beta}} \, \delta g_{\alpha\beta} = -\frac{1}{4\pi} \int d^2\sigma \sqrt{g} \, \epsilon \, T^\alpha{}_\alpha$$

But this must vanish in a conformal theory because scaling transformations are a symmetry. So

$$T^\alpha{}_\alpha = 0$$

This is the key feature of a conformal field theory in any dimension. Many theories have this feature at the classical level, including Maxwell theory and Yang-Mills theory in four-dimensions. However, it is much harder to preserve at the quantum level. (The weight of the world rests on the fact that Yang-Mills theory fails to be conformal at the quantum level). Technically the difficulty arises due to the need to introduce a scale when regulating the theories. Here we will be interested in two-dimensional theories

which succeed in preserving the conformal symmetry at the quantum level.

Looking Ahead: Even when the conformal invariance survives in a 2d quantum theory, the vanishing trace $T^\alpha_{\ \alpha} = 0$ will only turn out to hold in flat space. We will derive this result in section 4.4.2.

The Stress-Tensor in Complex Coordinates

In complex coordinates, $z = \sigma^1 + i\sigma^2$, the vanishing of the trace $T^\alpha_{\ \alpha} = 0$ becomes

$$T_{z\bar{z}} = 0$$

Meanwhile, the conservation equation $\partial_\alpha T^{\alpha\beta} = 0$ becomes $\partial T^{zz} = \bar{\partial} T^{\bar{z}\bar{z}} = 0$. Or, lowering the indices on T,

$$\bar{\partial}\, T_{zz} = 0 \quad \text{and} \quad \partial\, T_{\bar{z}\bar{z}} = 0$$

In other words, $T_{zz} = T_{zz}(z)$ is a holomorphic function while $T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})$ is an anti-holomorphic function. We will often use the simplified notation

$$T_{zz}(z) \equiv T(z) \quad \text{and} \quad T_{\bar{z}\bar{z}}(\bar{z}) \equiv \bar{T}(\bar{z})$$

4.1.2 Noether Currents

The stress-energy tensor $T_{\alpha\beta}$ provides the Noether currents for translations. What are the currents associated to the other conformal transformations? Consider the infinitesimal change,

$$z' = z + \epsilon(z) \quad , \quad \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$$

where, making contact with the two examples above, constant ϵ corresponds to a translation while $\epsilon(z) \sim z$ corresponds to a rotation and dilatation. To compute the current, we'll use the same trick that we saw before: we promote the parameter ϵ to depend on the worldsheet coordinates. But it's already a function of half of the worldsheet coordinates, so this now means $\epsilon(z) \to \epsilon(z, \bar{z})$. Then we can compute the change in the action, again using the fact that we can make a compensating change in the metric,

$$\begin{aligned}
\delta S &= -\int d^2\sigma \ \frac{\partial S}{\partial g^{\alpha\beta}}\, \delta g^{\alpha\beta} \\
&= \frac{1}{2\pi} \int d^2\sigma \ T_{\alpha\beta} \left(\partial^\alpha \delta\sigma^\beta\right) \\
&= \frac{1}{2\pi} \int d^2 z \ \frac{1}{2} \left[T_{zz} \left(\partial^z \delta z\right) + T_{\bar{z}\bar{z}} \left(\partial^{\bar{z}} \delta \bar{z}\right)\right] \\
&= \frac{1}{2\pi} \int d^2 z \ \left[T_{zz}\, \partial_{\bar{z}} \epsilon + T_{\bar{z}\bar{z}}\, \partial_z \bar{\epsilon}\right]
\end{aligned} \tag{4.6}$$

Firstly note that if ϵ is holomorphic and $\bar{\epsilon}$ is anti-holomorphic, then we immediately have $\delta S = 0$. This, of course, is the statement that we have a symmetry on our hands. (You may wonder where in the above derivation we used the fact that the theory was conformal. It lies in the transition to the third line where we needed $T_{z\bar{z}} = 0$).

At this stage, let's use the trick of treating z and $\bar{z}$ as independent variables. We look at separate currents that come from shifts in z and shifts $\bar{z}$. Let's first look at the symmetry

$$\delta z = \epsilon(z) \quad , \quad \delta\bar{z} = 0$$

We can read off the conserved current from (4.6) by using the standard trick of letting the small parameter depend on position. Since $\epsilon(z)$ already depends on position, this means promoting $\epsilon \to \epsilon(z)f(\bar{z})$ for some function f and then looking at the $\bar{\partial}f$ terms in (4.6). This gives us the current

$$J^z = 0 \quad \text{and} \quad J^{\bar{z}} = T_{zz}(z)\,\epsilon(z) \equiv T(z)\,\epsilon(z) \tag{4.7}$$

Importantly, we find that the current itself is also holomorphic. We can check that this is indeed a conserved current: it should satisfy $\partial_\alpha J^\alpha = \partial_z J^z + \partial_{\bar{z}} J^{\bar{z}} = 0$. But in fact it does so with room to spare: it satisfies the much stronger condition $\partial_{\bar{z}} J^{\bar{z}} = 0$.

Similarly, we can look at transformations $\delta\bar{z} = \bar{\epsilon}(\bar{z})$ with $\delta z = 0$. We get the anti-holomorphic current $\bar{J}$,

$$\bar{J}^z = \bar{T}(\bar{z})\,\bar{\epsilon}(\bar{z}) \quad \text{and} \quad \bar{J}^{\bar{z}} = 0 \tag{4.8}$$

4.1.3 An Example: The Free Scalar Field

Let's illustrate some of these ideas about classical conformal theories with the free scalar field,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha X\,\partial^\alpha X$$

Notice that there's no overall minus sign, in contrast to our earlier action (1.30). That's because we're now working with a Euclidean worldsheet metric. The theory of a free scalar field is, of course, dead easy. We can compute anything we like in this theory. Nonetheless, it will still exhibit enough structure to provide an example of all the abstract concepts that we will come across in CFT. For this reason, the free scalar field will prove a good companion throughout this part of the lectures.

Firstly, let's just check that this free scalar field is actually conformal. In particular, we can look at rescaling $\sigma^\alpha \to \lambda\sigma^\alpha$. If we view this in the sense of an active transformation, the coordinates remain fixed but the value of the field at point σ gets moved to point $\lambda\sigma$. This means,

$$X(\sigma) \;\to\; X(\lambda^{-1}\sigma) \quad \text{and} \quad \frac{\partial X(\sigma)}{\partial\sigma^\alpha} \;\to\; \frac{\partial X(\lambda^{-1}\sigma)}{\partial\sigma^\alpha} = \frac{1}{\lambda}\frac{\partial X(\tilde\sigma)}{\partial\tilde\sigma}$$

where we've defined $\tilde\sigma = \lambda^{-1}\sigma$. The factor of λ^{-2} coming from the two derivatives in the Lagrangian then cancels the Jacobian factor from the measure $d^2\sigma = \lambda^2\, d^2\tilde\sigma$, leaving the action invariant. Note that any polynomial interaction term for X would break conformal invariance.

The stress-energy tensor for this theory is defined using (4.4),

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\left(\partial_\alpha X\partial_\beta X - \frac{1}{2}\,\delta_{\alpha\beta}(\partial X)^2\right)\ , \tag{4.9}$$

which indeed satisfies $T^\alpha_{\ \alpha} = 0$ as it should. The stress-energy tensor looks much simpler in complex coordinates. It is simple to check that $T_{z\bar z} = 0$ while

$$T = -\frac{1}{\alpha'}\,\partial X\partial X \quad \text{and} \quad \bar T = -\frac{1}{\alpha'}\,\bar\partial X\bar\partial X$$

The equation of motion for X is $\partial\bar\partial X = 0$. The general classical solution decomposes as,

$$X(z,\bar z) = X(z) + \bar X(\bar z)$$

When evaluated on this solution, T and $\bar T$ become holomorphic and anti-holomorphic functions respectively.

4.2 Quantum Aspects

So far our discussion has been entirely classical. We now turn to the quantum theory. The first concept that we want to discuss is actually a feature of any quantum field theory. But it really comes into its own in the context of CFT: it is the *operator product expansion*.

4.2.1 Operator Product Expansion

Let's first describe what we mean by a *local* operator in a CFT. We will also refer to these objects as *fields*. There is a slight difference in terminology between CFTs and more general quantum field theories. Usually in quantum field theory, one reserves the

term "field" for the objects ϕ which sit in the action and are integrated over in the path integral. In contrast, in CFT the term "field" refers to any local expression that we can write down. This includes ϕ, but also includes derivatives $\partial^n\phi$ or composite operators such as $e^{i\phi}$. All of these are thought of as different fields in a CFT. It should be clear from this that the set of all "fields" in a CFT is always infinite even though, if you were used to working with quantum field theory, you would talk about only a finite number of fundamental objects ϕ. Obviously, this is nothing to be scared about. It's just a change of language: it doesn't mean that our theory got harder.

We now define the *operator product expansion* (OPE). It is a statement about what happens as local operators approach each other. The idea is that two local operators inserted at nearby points can be closely approximated by a string of operators at one of these points. Let's denote all the local operators of the CFT by $\mathcal{O}_i$, where i runs over the set of all operators. Then the OPE is

$$\mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w}) = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\,\mathcal{O}_k(w,\bar{w}) \tag{4.10}$$

Here $C_{ij}^k(z-w,\bar{z}-\bar{w})$ are a set of functions which, on grounds of translational invariance, depend only on the separation between the two operators. We will write a lot of operator equations of the form (4.10) and it's important to clarify exactly what they mean: they are always to be understood as statements which hold as operator insertions inside time-ordered correlation functions,

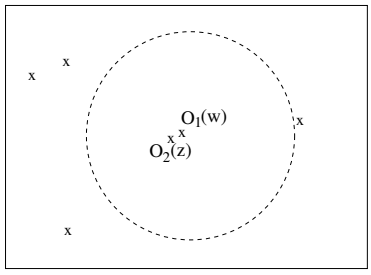


Figure 19:

$$\langle \mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w})\ldots\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\,\langle \mathcal{O}_k(w,\bar{w})\ldots\rangle$$

where the ... can be any other operator insertions that we choose. Obviously it would be tedious to continually write $\langle\ldots\rangle$. So we don't. But it's always implicitly there. There are further caveats about the OPE that are worth stressing

- The correlation functions are always assumed to be time-ordered. (Or something similar that we will discuss in Section 4.5.1). This means that as far as the OPE is concerned, everything commutes since the ordering of operators is determined inside the correlation function anyway. So we must have $\mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w}) = \mathcal{O}_j(w,\bar{w})\,\mathcal{O}_i(z,\bar{z})$. (There is a caveat here: if the operators are Grassmann objects, then they pick up an extra minus sign when commuted, even inside time-ordered products).

- The other operator insertions in the correlation function (denoted ... above) are arbitrary. *Except* they should be at a distance large compared to $|z-w|$. It turns out — rather remarkably — that in a CFT the OPEs are exact statements and have a radius of convergence equal to the distance to the nearest other insertion. We will return to this in Section 4.6. The radius of convergence is denoted in the figure by the dotted line.

- The OPEs have singular behaviour as $z \to w$. In fact, this singular behaviour will really be the only thing we care about! It will turn out to contain the same information as commutation relations, as well as telling us how operators transform under symmetries. Indeed, in many equations we will simply write the singular terms in the OPE and denote the non-singular terms as $+\ldots$.

4.2.2 Ward Identities

The spirit of Noether's theorem in quantum field theories is captured by operator equations known as *Ward Identities*. Here we derive the Ward identities associated to conformal invariance. We start by considering a general theory with a symmetry. Later we will restrict to conformal symmetries.

Games with Path Integrals

We'll take this opportunity to get comfortable with some basic techniques using path integrals. Schematically, the path integral takes the form

$$Z = \int \mathcal{D}\phi \; e^{-S[\phi]}$$

where ϕ collectively denote all the fields (in the path integral sense...not the CFT sense!). A symmetry of the quantum theory is such that an infinitesimal transformation

$$\phi' = \phi + \epsilon\delta\phi$$

leaves both the action *and* the measure invariant,

$$S[\phi'] = S[\phi] \quad \text{and} \quad \mathcal{D}\phi' = \mathcal{D}\phi$$

(In fact, we only really need the combination $\mathcal{D}\phi\, e^{-S[\phi]}$ to be invariant but this subtlety won't matter in this course). We use the same trick that we employed earlier in the classical theory and promote $\epsilon \to \epsilon(\sigma)$. Then, typically, neither the action nor the measure are invariant but, to leading order in ϵ, the change has to be proportional to

$\partial\epsilon$. We have

$$\begin{aligned} Z &\longrightarrow \int \mathcal{D}\phi' \ \exp\left(-S[\phi']\right) \\ &= \int \mathcal{D}\phi \ \exp\left(-S[\phi] - \frac{1}{2\pi}\int J^\alpha \, \partial_\alpha \epsilon\right) \\ &= \int \mathcal{D}\phi \ e^{-S[\phi]}\left(1 - \frac{1}{2\pi}\int J^\alpha \, \partial_\alpha \epsilon\right) \end{aligned}$$

where the factor of $1/2\pi$ is merely a convention and $\int$ is shorthand for $\int d^2\sigma\sqrt{g}$. Notice that the current J^α may now also have contributions from the measure transformation as well as the action.

Now comes the clever step. Although the integrand has changed, the actual value of the partition function can't have changed at all. After all, we just redefined a dummy integration variable ϕ. So the expression above must be equal to the original Z. Or, in other words,

$$\int \mathcal{D}\phi \, e^{-S[\phi]} \ \left(\int J^\alpha \, \partial_\alpha \epsilon\right) = 0$$

Moreover, this must hold for all ϵ. This gives us the quantum version of Noether's theorem: the vacuum expectation value of the divergence of the current vanishes:

$$\langle \partial_\alpha J^\alpha \rangle = 0 \ .$$

We can repeat these tricks of this sort to derive some stronger statements. Let's see what happens when we have other insertions in the path integral. The time-ordered correlation function is given by

$$\langle \mathcal{O}_1(\sigma_1) \ldots \mathcal{O}_n(\sigma_n) \rangle = \frac{1}{Z}\int \mathcal{D}\phi \, e^{-S[\phi]} \, \mathcal{O}_1(\sigma_1) \ldots \mathcal{O}_n(\sigma_n)$$

We can think of these as operators inserted at particular points on the plane as shown in the figure. As we described above, the operators $\mathcal{O}_i$ are any general expressions that we can form from the ϕ fields. Under the symmetry of interest, the operator will change in some way, say

$$\mathcal{O}_i \to \mathcal{O}_i + \epsilon \, \delta\mathcal{O}_i$$

We once again promote $\epsilon \to \epsilon(\sigma)$. As our first pass, let's pick a choice of $\epsilon(\sigma)$ which only has support away from the operator insertions as shown in the Figure 20. Then,

$$\delta\mathcal{O}_i(\sigma_i) = 0$$

and the above derivation goes through in exactly the same way to give

$$\langle \partial_\alpha J^\alpha(\sigma)\, \mathcal{O}_1(\sigma_1) \ldots \mathcal{O}_n(\sigma_n) \rangle = 0 \qquad \text{for } \sigma \neq \sigma_i$$

Because this holds for any operator insertions away from σ, from the discussion in Section 4.2.1 we are entitled to write the operator equation

$$\partial_\alpha J^\alpha = 0$$

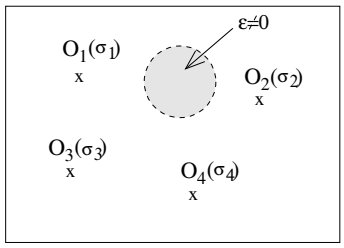


Figure 20:

But what if there are operator insertions that lie at the same point as J^α? In other words, what happens as σ approaches one of the insertion points? The resulting formulae are called Ward identities. To derive these, let's take $\epsilon(\sigma)$ to have support in some region that includes the point σ_1, but not the other points as shown in Figure 21. The simplest choice is just to take $\epsilon(\sigma)$ to be constant inside the shaded region and zero outside. Now using the same procedure as before, we find that the original correlation function is equal to,

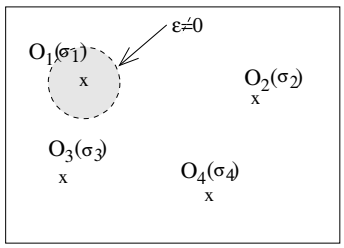


Figure 21:

$$\frac{1}{Z} \int \mathcal{D}\phi\, e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int J^\alpha\, \partial_\alpha \epsilon\right) (\mathcal{O}_1 + \epsilon\, \delta\mathcal{O}_1)\, \mathcal{O}_2 \ldots \mathcal{O}_n$$

Working to leading order in ϵ, this gives

$$-\frac{1}{2\pi} \int_\epsilon \partial_\alpha \langle J^\alpha(\sigma)\, \mathcal{O}_1(\sigma_1) \ldots \rangle = \langle \delta\mathcal{O}_1(\sigma_1) \ldots \rangle \tag{4.11}$$

where the integral on the left-hand-side is only over the region of non-zero ϵ. This is the *Ward Identity*.

Ward Identities for Conformal Transformations

Ward identities (4.11) hold for any symmetries. Let's now see what they give when applied to conformal transformations. There are two further steps needed in the derivation. The first simply comes from the fact that we're working in two dimensions and we can use Stokes' theorem to convert the integral on the left-hand-side of (4.11) to a line integral around the boundary. Let $\hat{n}^\alpha$ be the unit vector normal to the boundary. For any vector J^α, we have

$$\int_\epsilon \partial_\alpha J^\alpha = \oint_{\partial\epsilon} J_\alpha \hat{n}^\alpha = \oint_{\partial\epsilon} (J_1\, d\sigma^2 - J_2\, d\sigma^1) = -i \oint_{\partial\epsilon} (J_z\, dz - J_{\bar{z}}\, d\bar{z})$$

where we have written the expression both in Cartesian coordinates σ^α and complex coordinates on the plane. As described in Section 4.0.1, the complex components of the vector with indices down are defined as $J_z = \frac{1}{2}(J_1 - iJ_2)$ and $J_{\bar{z}} = \frac{1}{2}(J_1 + iJ_2)$. So, applying this to the Ward identity (4.11), we find for two dimensional theories

$$\frac{i}{2\pi}\oint_{\partial\epsilon} dz\ \langle J_z(z,\bar{z})\,\mathcal{O}_1(\sigma_1)\ldots\rangle - \frac{i}{2\pi}\oint_{\partial\epsilon} d\bar{z}\ \langle J_{\bar{z}}(z,\bar{z})\,\mathcal{O}_1(\sigma_1)\ldots\rangle = \langle\delta\mathcal{O}_1(\sigma_1)\ldots\rangle$$

So far our derivation holds for any conserved current J in two dimensions. At this stage we specialize to the currents that arise from conformal transformations (4.7) and (4.8). Here something nice happens because J_z is holomorphic while $J_{\bar{z}}$ is anti-holomorphic. This means that the contour integral simply picks up the residue,

$$\frac{i}{2\pi}\oint_{\partial\epsilon} dz\, J_z(z)\mathcal{O}_1(\sigma_1) = -\,\mathrm{Res}\,[J_z\mathcal{O}_1]$$

where this means the residue in the OPE between the two operators,

$$J_z(z)\,\mathcal{O}_1(w,\bar{w}) = \ldots + \frac{\mathrm{Res}\,[J_z\mathcal{O}_1(w,\bar{w})]}{z-w} + \ldots$$

So we find a rather nice way of writing the Ward identities for conformal transformations. If we again view z and $\bar{z}$ as independent variables, the Ward identities split into two pieces. From the change $\delta z = \epsilon(z)$, we get

$$\delta\mathcal{O}_1(\sigma_1) = -\mathrm{Res}\,[J_z(z)\mathcal{O}_1(\sigma_1)] = -\mathrm{Res}\,[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)] \tag{4.12}$$

where, in the second equality, we have used the expression for the conformal current (4.7). Meanwhile, from the change $\delta\bar{z} = \bar{\epsilon}(\bar{z})$, we have

$$\delta\mathcal{O}_1(\sigma_1) = -\mathrm{Res}\,[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_1(\sigma_1)] = -\mathrm{Res}\,[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_1(\sigma_1)]$$

where the minus sign comes from the fact that the $\oint d\bar{z}$ boundary integral is taken in the opposite direction.

This result means that if we know the OPE between an operator and the stress-tensors $T(z)$ and $\bar{T}(\bar{z})$, then we immediately know how the operator transforms under conformal symmetry. Or, standing this on its head, if we know how an operator transforms then we know at least some part of its OPE with T and $\bar{T}$.

4.2.3 Primary Operators

The Ward identity allows us to start piecing together some OPEs by looking at how operators transform under conformal symmetries. Although we don't yet know the

action of general conformal symmetries, we can start to make progress by looking at the two simplest examples.

Translations: If $\delta z = \epsilon$, a constant, then all operators transform as

$$\mathcal{O}(z - \epsilon) = \mathcal{O}(z) - \epsilon\, \partial\mathcal{O}(z) + \dots$$

The Noether current for translations is the stress-energy tensor T. The Ward identity in the form (4.12) tells us that the OPE of T with any operator $\mathcal{O}$ must be of the form,

$$T(z)\, \mathcal{O}(w, \bar{w}) = \dots + \frac{\partial\mathcal{O}(w, \bar{w})}{z - w} + \dots \tag{4.13}$$

Similarly, the OPE with $\bar{T}$ is

$$\bar{T}(\bar{z})\, \mathcal{O}(w, \bar{w}) = \dots + \frac{\bar{\partial}\mathcal{O}(w, \bar{w})}{\bar{z} - \bar{w}} + \dots \tag{4.14}$$

Rotations and Scaling: The transformation

$$z \to z + \epsilon z \quad \text{and} \quad \bar{z} \to \bar{z} + \bar{\epsilon}\bar{z} \tag{4.15}$$

describes rotation for ϵ purely imaginary and scaling (dilatation) for ϵ real. Not all operators have good transformation properties under these actions. This is entirely analogous to the statement in quantum mechanics that not all states transform nicely under the Hamiltonian H and angular momentum operator L. However, in quantum mechanics we know that the eigenstates of H and L can be chosen as a basis of the Hilbert space provided, of course, that $[H, L] = 0$.

The same statement holds for operators in a CFT: we can choose a basis of local operators that have good transformation properties under rotations and dilatations. In fact, we will see in Section 4.6 that the statement about local operators actually follows from the statement about states.

Definition: An operator $\mathcal{O}$ is said to have *weight* $(h, \tilde{h})$ if, under $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}$, $\mathcal{O}$ transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\, \partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\, \bar{\partial}\mathcal{O}) \tag{4.16}$$

The terms $\partial\mathcal{O}$ in this expression would be there for any operator. They simply come from expanding $\mathcal{O}(z - \epsilon z, \bar{z} - \bar{\epsilon}\bar{z})$. The terms $h\mathcal{O}$ and $\tilde{h}\mathcal{O}$ are special to operators which are eigenstates of dilatations and rotations. Some comments:

- Both h and $\tilde{h}$ are real numbers. In a unitary CFT, all operators have $h, \tilde{h} \geq 0$. We will prove this is Section 4.5.4.

- The weights are not as unfamiliar as they appear. They simply tell us how operators transform under rotations and scalings. But we already have names for these concepts from undergraduate days. The eigenvalue under rotation is usually called the *spin*, s, and is given in terms of the weights as

  $$s = h - \tilde{h}$$

  Meanwhile, the *scaling dimension* Δ of an operator is

  $$\Delta = h + \tilde{h}$$

- To motivate these definitions, it's worth recalling how rotations and scale transformations act on the underlying coordinates. Rotations are implemented by the operator

  $$L = -i(\sigma^1\partial_2 - \sigma^2\partial_1) = z\partial - \bar{z}\bar{\partial}$$

  while the dilation operator D which gives rise to scalings is

  $$D = \sigma^\alpha\partial_\alpha = z\partial + \bar{z}\bar{\partial}$$

- The scaling dimension is nothing more than the familiar "dimension" that we usually associate to fields and operators by dimensional analysis. For example, worldsheet derivatives always increase the dimension of an operator by one: $\Delta[\partial] = +1$. The tricky part is that the naive dimension that fields have in the classical theory is not necessarily the same as the dimension in the quantum theory.

Let's compare the transformation law (4.16) with the Ward identity (4.12). The Noether current arising from rotations and scaling $\delta z = \epsilon z$ was given in (4.7): it is $J(z) = zT(z)$. This means that the residue of the $J\mathcal{O}$ OPE will determine the $1/z^2$ term in the $T\mathcal{O}$ OPE. Similar arguments hold, of course, for $\delta\bar{z} = \bar{\epsilon}\bar{z}$ and $\bar{T}$. So, the upshot of this is that, for an operator $\mathcal{O}$ with weight $(h, \tilde{h})$, the OPE with T and $\bar{T}$ takes the form

$$T(z)\,\mathcal{O}(w,\bar{w}) = \ldots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots$$
$$\bar{T}(\bar{z})\,\mathcal{O}(w,\bar{w}) = \ldots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \ldots$$

Primary Operators

A *primary* operator is one whose OPE with T and $\bar{T}$ truncates at order $(z-w)^{-2}$ or order $(\bar{z}-\bar{w})^{-2}$ respectively. There are no higher singularities:

$$T(z)\,\mathcal{O}(w,\bar{w}) = h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \text{non-singular}$$
$$\bar{T}(\bar{z})\,\mathcal{O}(w,\bar{w}) = \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \text{non-singular}$$

Since we now know all singularities in the $T\mathcal{O}$ OPE, we can reconstruct the transformation under all conformal transformations. The importance of primary operators is that they have particularly simple transformation properties. Focussing on $\delta z = \epsilon(z)$, we have

$$\begin{aligned}\delta\mathcal{O}(w,\bar{w}) &= -\text{Res}\left[\epsilon(z)\,T(z)\,\mathcal{O}(w,\bar{w})\right] \\ &= -\text{Res}\left[\epsilon(z)\left(h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots\right)\right]\end{aligned}$$

We want to look at smooth conformal transformations and so require that $\epsilon(z)$ itself has no singularities at $z=w$. We can then Taylor expand

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)\,(z-w) + \ldots$$

We learn that the infinitesimal change of a primary operator under a general conformal transformation $\delta z = \epsilon(z)$ is

$$\delta\mathcal{O}(w,\bar{w}) = -h\epsilon'(w)\,\mathcal{O}(w,\bar{w}) - \epsilon(w)\,\partial\mathcal{O}(w,\bar{w}) \tag{4.17}$$

There is a similar expression for the anti-holomorphic transformations $\delta\bar{z} = \bar{\epsilon}(\bar{z})$.

Equation (4.17) holds for infinitesimal conformal transformations. It is a simple matter to integrate up to find how primary operators change under a finite conformal transformation,

$$z \to \tilde{z}(z) \quad \text{and} \quad \bar{z} \to \bar{\tilde{z}}(\bar{z})$$

The general transformation of a primary operator is given by

$$\mathcal{O}(z,\bar{z}) \;\to\; \tilde{\mathcal{O}}(\tilde{z},\bar{\tilde{z}}) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-h}\left(\frac{\partial\bar{\tilde{z}}}{\partial\bar{z}}\right)^{-\tilde{h}}\mathcal{O}(z,\bar{z}) \tag{4.18}$$

It will turn out that one of the main objects of interest in a CFT is the spectrum of weights $(h,\tilde{h})$ of primary fields. This will be equivalent to computing the particle mass spectrum in a quantum field theory. In the context of statistical mechanics, the weights of primary operators are the critical exponents.

4.3 An Example: The Free Scalar Field

Let's look at how all of this works for the free scalar field. We'll start by familiarizing ourselves with some techniques using the path integral. The action is,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \; \partial_\alpha X \, \partial^\alpha X \tag{4.19}$$

The classical equation of motion is $\partial^2 X = 0$. Let's start by seeing how to derive the analogous statement in the quantum theory using the path integral. The key fact that we'll need is that the integral of a total derivative vanishes in the path integral just as it does in an ordinary integral. From this we have,

$$0 = \int \mathcal{D}X \; \frac{\delta}{\delta X(\sigma)} \, e^{-S} = \int \mathcal{D}X \; e^{-S} \left[\frac{1}{2\pi\alpha'} \, \partial^2 X(\sigma) \right]$$

But this is nothing more than the Ehrenfest theorem which states that expectation values of operators obey the classical equations of motion,

$$\langle \partial^2 X(\sigma) \rangle = 0$$

4.3.1 The Propagator

The next thing that we want to do is compute the propagator for X. We could do this using canonical quantization, but it will be useful to again see how it works using the path integral. This time we look at,

$$0 = \int \mathcal{D}X \; \frac{\delta}{\delta X(\sigma)} \left[e^{-S} \, X(\sigma') \right] = \int \mathcal{D}X \; e^{-S} \left[\tfrac{1}{2\pi\alpha'} \, \partial^2 X(\sigma) \, X(\sigma') + \delta(\sigma - \sigma') \right]$$

So this time we learn that

$$\langle \partial^2 X(\sigma) \, X(\sigma') \rangle = -2\pi\alpha' \, \delta(\sigma - \sigma') \tag{4.20}$$

Note that if we'd computed this in the canonical approach, we would have found the same answer: the δ-function arises in this calculation because all correlation functions are time-ordered.

We can now treat (4.20) as a differential equation for the propagator $\langle X(\sigma) X(\sigma') \rangle$. To solve this equation, we need the following standard result

$$\partial^2 \ln(\sigma - \sigma')^2 = 4\pi\delta(\sigma - \sigma') \tag{4.21}$$

Since this is important, let's just quickly check that it's true. It's a simple application of Stokes' theorem. Set $\sigma' = 0$ and integrate over $\int d^2\sigma$. We obviously get 4π from the right-hand-side. The left-hand-side gives

$$\int d^2\sigma\ \partial^2 \ln(\sigma_1^2 + \sigma_2^2) = \int d^2\sigma\ \partial^\alpha \left(\frac{2\sigma_\alpha}{\sigma_1^2 + \sigma_2^2}\right) = 2 \oint \frac{(\sigma_1\, d\sigma^2 - \sigma_2 d\sigma^1)}{\sigma_1^2 + \sigma_2^2}$$

Switching to polar coordinates $\sigma_1 + i\sigma_2 = re^{i\theta}$, we can rewrite this expression as

$$2\int \frac{r^2 d\theta}{r^2} = 4\pi$$

confirming (4.21). Applying this result to our equation (4.20), we get the propagator of a free scalar in two-dimensions,

$$\langle X(\sigma)X(\sigma')\rangle = -\frac{\alpha'}{2}\ln(\sigma - \sigma')^2$$

The propagator has a singularity as $\sigma \to \sigma'$. This is an ultra-violet divergence and is common to all field theories. It also has a singularity as $|\sigma - \sigma'| \to \infty$. This is telling us something important that we will mention in Section 4.3.2.

Finally, we could repeat our trick of looking at total derivatives in the path integral, now with other operator insertions $\mathcal{O}_1(\sigma_1), \ldots \mathcal{O}_n(\sigma_n)$ in the path integral. As long as $\sigma, \sigma' \neq \sigma_i$, then the whole analysis goes through as before. But this is exactly our criterion to write the operator product equation,

$$X(\sigma)X(\sigma') = -\frac{\alpha'}{2}\ln(\sigma - \sigma')^2 + \ldots \tag{4.22}$$

We can also write this in complex coordinates. The classical equation of motion $\partial\bar{\partial}X = 0$ allows us to split the operator X into left-moving and right-moving pieces,

$$X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$$

We'll focus just on the left-moving piece. This has the operator product expansion,

$$X(z)X(w) = -\frac{\alpha'}{2}\ln(z - w) + \ldots$$

The logarithm means that $X(z)$ doesn't have any nice properties under the conformal transformations. For this reason, the "fundamental field" X is not really the object of interest in this theory! However, we can look at the derivative of X. This has a rather nice looking OPE,

$$\partial X(z)\, \partial X(w) = -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + \text{non-singular} \tag{4.23}$$

4.3.2 An Aside: No Goldstone Bosons in Two Dimensions

The infra-red divergence in the propagator has an important physical implication. Let's start by pointing out one of the big differences between quantum mechanics and quantum field theory in $d = 3 + 1$ dimensions. Since the language used to describe these two theories is rather different, you may not even be aware that this difference exists.

Consider the quantum mechanics of a particle on a line. This is a $d = 0 + 1$ dimensional theory of a free scalar field X. Let's prepare the particle in some localized state – say a Gaussian wavefunction $\Psi(X) \sim \exp(-X^2/L^2)$. What then happens? The wavefunction starts to spread out. And the spreading doesn't stop. In fact, the would-be ground state of the system is a uniform wavefunction of infinite width, which isn't a state in the Hilbert space because it is non-normalizable.

Let's now compare this to the situation of a free scalar field X in a $d = 3 + 1$ dimensional field theory. Now we think of this as a scalar without potential. The physics is very different: the theory has an infinite number of ground states, determined by the expectation value $\langle X \rangle$. Small fluctuations around this vacuum are massless: they are Goldstone bosons for broken translational invariance $X \to X + c$.

We see that the physics is very different in field theories in $d = 0 + 1$ and $d = 3 + 1$ dimensions. The wavefunction spreads along flat directions in quantum mechanics, but not in higher dimensional field theories. But what happens in $d = 1 + 1$ and $d = 2 + 1$ dimensions? It turns out that field theories in $d = 1 + 1$ dimensions are more like quantum mechanics: the wavefunction spreads. Theories in $d = 2 + 1$ dimensions and higher exhibit the opposite behaviour: they have Goldstone bosons. The place to see this is the propagator. In d spacetime dimensions, it takes the form

$$\langle X(r)\, X(0) \rangle \sim \begin{cases} 1/r^{d-2} & d \neq 2 \\ \ln r & d = 2 \end{cases}$$

which diverges at large r only for $d = 1$ and $d = 2$. If we perturb the vacuum slightly by inserting the operator $X(0)$, this correlation function tells us how this perturbation falls off with distance. The infra-red divergence in low dimensions is telling us that the wavefunction wants to spread.

The spreading of the wavefunction in low dimensions means that there is no spontaneous symmetry breaking and no Goldstone bosons. It is usually referred to as the Coleman-Mermin-Wagner theorem. Note, however, that it certainly doesn't prohibit massless excitations in two dimensions: it only prohibits Goldstone-like massless excitations.

4.3.3 The Stress-Energy Tensor and Primary Operators

We want to compute the OPE of T with other operators. Firstly, what is T? We computed it in the classical theory in (4.9). It is,

$$T = -\frac{1}{\alpha'}\,\partial X \partial X \tag{4.24}$$

But we need to be careful about what this means in the quantum theory. It involves the product of two operators defined at the same point and this is bound to mean divergences if we just treat it naively. In canonical quantization, we would be tempted to normal order by putting all annihilation operators to the right. This guarantees that the vacuum has zero energy. Here we do something that is basically equivalent, but without reference to creation and annihilation operators. We write

$$T = -\frac{1}{\alpha'}\,:\partial X \partial X: \;\equiv\; -\frac{1}{\alpha'}\lim_{z\to w}\left(\partial X(z)\partial X(w) - \langle \partial X(z)\partial X(w)\rangle\right) \tag{4.25}$$

which, by construction, has $\langle T\rangle = 0$.

With this definition of T, let's start to compute the OPEs to determine the primary fields in the theory.

Claim 1: ∂X is a primary field with weight $h = 1$ and $\tilde{h} = 0$.

Proof: We need to figure out how to take products of normal ordered operators

$$T(z)\,\partial X(w) = -\frac{1}{\alpha'}\,:\partial X(z)\partial X(z):\partial X(w)$$

The operators on the left-hand side are time-ordered (because all operator expressions of this type are taken to live inside time-ordered correlation functions). In contrast, the right-hand side is a product of normal-ordered operators. But we know how to change normal ordered products into time ordered products: this is the content of Wick's theorem. Although we have defined normal ordering in (4.25) without reference to creation and annihilation operators, Wick's theorem still holds. We must sum over all possible contractions of pairs of operators, where the term "contraction" means that we replace the pair by the propagator,

$$\overbrace{\partial X(z)\,\partial X(w)} = -\frac{\alpha'}{2}\,\frac{1}{(z-w)^2}$$

Using this, we have

$$T(z)\partial X(w) = -\frac{2}{\alpha'}\,\partial X(z)\,\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2} + \text{non-singular}\right)$$

Here the "non-singular" piece includes the totally normal ordered term $:T(z)\partial X(w):$. It is only the singular part that interests us. Continuing, we have

$$T(z)\partial X(w) = \frac{\partial X(z)}{(z-w)^2} + \ldots = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \ldots$$

This is indeed the OPE for a primary operator of weight $h = 1$. □

Note that higher derivatives $\partial^n X$ are not primary for $n > 1$. For example, $\partial^2 X$ has weight $(h, \tilde{h}) = (2, 0)$, but is not a primary operator, as we see from the OPE,

$$T(z)\,\partial^2 X(w) = \partial_w \left[\frac{\partial X(w)}{(z-w)^2} + \ldots \right] = \frac{2\partial X(w)}{(z-w)^3} + \frac{2\partial^2 X(w)}{(z-w)^2} + \ldots$$

The fact that the field $\partial^n X$ has weight $(h, \tilde{h}) = (n, 0)$ fits our natural intuition: each derivative provides spin $s = 1$ and dimension $\Delta = 1$, while the field X does not appear to be contributing, presumably reflecting the fact that it has naive, classical dimension zero. However, in the quantum theory, it is not correct to say that X has vanishing dimension: it has an ill-defined dimension due to the logarithmic behaviour of its OPE (4.22). This is responsible for the following, more surprising, result

Claim 2: The field $:e^{ikX}:$ is primary with weight $h = \tilde{h} = \alpha' k^2/4$.

This result is not what we would guess from the classical theory. Indeed, it's obvious that it has a quantum origin because the weight is proportional to α', which sits outside the action in the same place that $\hbar$ would (if we hadn't set it to one). Note also that this means that the spectrum of the free scalar field is continuous. This is related to the fact that the range of X is non-compact. Generally, CFTs will have a discrete spectrum.

Proof: Let's first compute the OPE with ∂X. We have

$$\begin{aligned}
\partial X(z)\ :e^{ikX(w)}: &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}\,\partial X(z)\ :X(w)^n: \\
&= \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!}\ :X(w)^{n-1}:\ \left(-\frac{\alpha'}{2}\,\frac{1}{z-w}\right) + \ldots \\
&= -\frac{i\alpha' k}{2}\,\frac{:e^{ikX(w)}:}{z-w} + \ldots
\end{aligned} \tag{4.26}$$

From this, we can compute the OPE with T.

$$\begin{aligned}
T(z)\ :e^{ikX(w)}: &= -\frac{1}{\alpha'}\ :\partial X(z)\partial X(z):\,:e^{ikX(w)}: \\
&= \frac{\alpha' k^2}{4}\,\frac{:e^{ikX(w)}:}{(z-w)^2} + ik\,\frac{:\partial X(z) e^{ikX(w)}:}{z-w} + \ldots
\end{aligned}$$

where the first term comes from two contractions, while the second term comes from a single contraction. Replacing ∂_z by ∂_w in the final term we get

$$T(z) \, : e^{ikX(w)} := \frac{\alpha' k^2}{4} \frac{: e^{ikX(w)} :}{(z-w)^2} + \frac{\partial_w : e^{ikX(w)} :}{z-w} + \dots \tag{4.27}$$

showing that $: e^{ikX(w)} :$ is indeed primary. We will encounter this operator frequently later, but will choose to simplify notation and drop the normal ordering colons. Normal ordering will just be assumed from now on. □.

Finally, lets check to see the OPE of T with itself. This is again just an exercise in Wick contractions.

$$\begin{aligned} T(z)\,T(w) &= \frac{1}{\alpha'^2} \; : \partial X(z)\, \partial X(z) : : \partial X(w)\, \partial X(w) : \\ &= \frac{2}{\alpha'^2} \left(-\frac{\alpha'}{2} \frac{1}{(z-w)^2} \right)^2 - \frac{4}{\alpha'^2} \frac{\alpha'}{2} \frac{: \partial X(z)\, \partial X(w) :}{(z-w)^2} + \dots \end{aligned}$$

The factor of 2 in front of the first term comes from the two ways of performing two contractions; the factor of 4 in the second term comes from the number of ways of performing a single contraction. Continuing,

$$\begin{aligned} T(z)\,T(w) &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} - \frac{2}{\alpha'} \frac{\partial^2 X(w)\, \partial X(w)}{z-w} + \dots \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \end{aligned} \tag{4.28}$$

We learn that T is *not* a primary operator in the theory of a single free scalar field. It is an operator of weight $(h, \tilde{h}) = (2, 0)$, but it fails the primary test on account of the $(z-w)^{-4}$ term. In fact, this property of the stress energy tensor a general feature of all CFTs which we now explore in more detail.

4.4 The Central Charge

In any CFT, the most prominent example of an operator which is not primary is the stress-energy tensor itself.

For the free scalar field, we have already seen that T is an operator of weight $(h, \tilde{h}) = (2, 0)$. This remains true in any CFT. The reason for this is simple: $T_{\alpha\beta}$ has dimension $\Delta = 2$ because we obtain the energy by integrating over space. It has spin $s = 2$ because it is a symmetric 2-tensor. But these two pieces of information are equivalent

to the statement that T is has weight $(2,0)$. Similarly, $\bar{T}$ has weight $(0,2)$. This means that the TT OPE takes the form,

$$T(z)\,T(w) = \ldots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

and similar for $\bar{T}\bar{T}$. What other terms could we have in this expansion? Since each term has dimension $\Delta = 4$, any operators that appear on the right-hand-side must be of the form

$$\frac{\mathcal{O}_n}{(z-w)^n} \tag{4.29}$$

where $\Delta[\mathcal{O}_n] = 4 - n$. But, in a unitary CFT there are no operators with $h,\, \tilde{h} < 0$. (We will prove this shortly). So the most singular term that we can have is of order $(z-w)^{-4}$. Such a term must be multiplied by a constant. We write,

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

and, similarly,

$$\bar{T}(\bar{z})\,\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \ldots$$

The constants c and $\tilde{c}$ are called the *central charges*. (Sometimes they are referred to as left-moving and right-moving central charges). They are perhaps the most important numbers characterizing the CFT. We can already get some intuition for the information contained in these two numbers. Looking back at the free scalar field (4.28) we see that it has $c = \tilde{c} = 1$. If we instead considered D non-interacting free scalar fields, we would get $c = \tilde{c} = D$. This gives us a hint: c and $\tilde{c}$ are somehow measuring the number of degrees of freedom in the CFT. This is true in a deep sense! However, be warned: c is not necessarily an integer.

Before moving on, it's worth pausing to explain why we didn't include a $(z-w)^{-3}$ term in the TT OPE. The reason is that the OPE must obey $T(z)T(w) = T(w)T(z)$ because, as explained previously, these operator equations are all taken to hold inside time-ordered correlation functions. So the quick answer is that a $(z-w)^{-3}$ term would not be invariant under $z \leftrightarrow w$. However, you may wonder how the $(z-w)^{-1}$ term manages to satisfy this property. Let's see how this works:

$$T(w)\,T(z) = \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{w-z} + \ldots$$

Now we can Taylor expand $T(z) = T(w) + (z-w)\partial T(w) + \ldots$ and $\partial T(z) = \partial T(w) + \ldots$. Using this in the above expression, we find

$$T(w)\,T(z) = \frac{c/2}{(z-w)^4} + \frac{2T(w) + 2(z-w)\partial T(w)}{(z-w)^2} - \frac{\partial T(w)}{z-w} + \ldots = T(z)\,T(w)$$

This trick of Taylor expanding saves the $(z-w)^{-1}$ term. It wouldn't work for the $(z-w)^{-3}$ term.

The Transformation of Energy

So T is not primary unless $c = 0$. And we will see shortly that all theories have $c > 0$. What does this mean for the transformation of T?

$$\begin{aligned}\delta T(w) &= -\text{Res}\left[\epsilon(z)\,T(z)\,T(w)\right] \\ &= -\text{Res}\left[\epsilon(z)\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots\right)\right]\end{aligned}$$

If $\epsilon(z)$ contains no singular terms, we can expand

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)(z-w) + \frac{1}{2}\epsilon''(z-w)^2 + \frac{1}{6}\epsilon'''(w)(z-w)^3 + \ldots$$

from which we find

$$\delta T(w) = -\epsilon(w)\,\partial T(w) - 2\epsilon'(w)\,T(w) - \frac{c}{12}\epsilon'''(w) \tag{4.30}$$

This is the infinitesimal version. We would like to know what becomes of T under the finite conformal transformation $z \to \tilde{z}(z)$. The answer turns out to be

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}\,S(\tilde{z}, z)\right] \tag{4.31}$$

where $S(\tilde{z}, z)$ is known as the *Schwarzian* and is defined by

$$S(\tilde{z}, z) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \tag{4.32}$$

It is simple to check that the Schwarzian has the right infinitesimal form to give (4.30). Its key property is that it preserves the group structure of successive conformal transformations.

4.4.1 c is for Casimir

Note that the extra term in the transformation (4.31) of T does not depend on T itself. In particular, it will be the same evaluated on all states. It only affects the constant term — or zero mode — in the energy. In other words, it is the Casimir energy of the system.

Let's look at an example that will prove to be useful later for the string. Consider the Euclidean cylinder, parameterized by

$$w = \sigma + i\tau \qquad , \qquad \sigma \in [0, 2\pi)$$

We can make a conformal transformation from the cylinder to the complex plane by

$$z = e^{-iw}$$

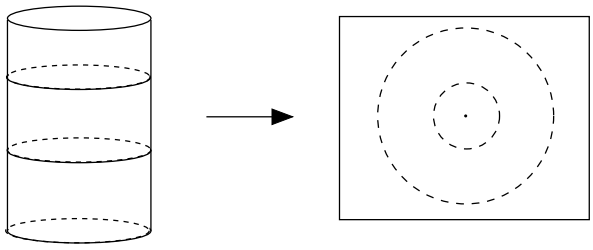


Figure 22:

The fact that the cylinder and the plane are related by a conformal map means that if we understand a given CFT on the cylinder, then we immediately understand it on the plane. And vice-versa. Notice that constant time slices on the cylinder are mapped to circles of constant radius. The origin, $z = 0$, is the distant past, $\tau \to -\infty$.

What becomes of T under this transformation? The Schwarzian can be easily calculated to be $S(z, w) = 1/2$. So we find,

$$T_{\text{cylinder}}(w) = -z^2\, T_{\text{plane}}(z) + \frac{c}{24} \tag{4.33}$$

Suppose that the ground state energy vanishes when the theory is defined on the plane: $\langle T_{\text{plane}} \rangle = 0$. What happens on the cylinder? We want to look at the Hamiltonian, which is defined by

$$H \equiv \int d\sigma\; T_{\tau\tau} = -\int d\sigma\, (T_{ww} + \bar{T}_{\bar{w}\bar{w}})$$

The conformal transformation then tells us that the ground state energy on the cylinder is

$$E = -\frac{2\pi(c + \tilde{c})}{24}$$

This is indeed the (negative) Casimir energy on a cylinder. For a free scalar field, we have $c = \tilde{c} = 1$ and the energy density $E/2\pi = -1/12$. This is the same result that we got in Section 2.2.2, but this time with no funny business where we throw out infinities.

An Application: The Lüscher Term

If we're looking at a physical system, the cylinder will have a radius L. In this case, the Casimir energy is given by $E = -2\pi(c+\tilde{c})/24L$. There is an application of this to QCD-like theories. Consider two quarks in a confining theory, separated by a distance L. If the tension of the confining flux tube is T, then the string will be stable as long as $TL \lesssim m$, the mass of the lightest quark. The energy of the stretched string as a function of L is given by

$$E(L) = TL + a - \frac{\pi c}{24L} + \ldots$$

Here a is an undetermined constant, while c counts the number of degrees of freedom of the QCD flux tube. (There is no analog of $\tilde{c}$ here because of the reflecting boundary conditions at the end of the string). If the string has no internal degrees of freedom, then $c = 2$ for the two transverse fluctuations. This contribution to the string energy is known as the *Lüscher term*.

4.4.2 The Weyl Anomaly

There is another way in which the central charge affects the stress-energy tensor. Recall that in the classical theory, one of the defining features of a CFT was the vanishing of the trace of the stress tensor,

$$T^{\alpha}_{\ \alpha} = 0$$

However, things are more subtle in the quantum theory. While $\langle T^{\alpha}_{\ \alpha}\rangle$ indeed vanishes in flat space, it will not longer be true if we place the theory on a curved background. The purpose of this section is to show that

$$\langle T^{\alpha}_{\ \alpha}\rangle = -\frac{c}{12}R \tag{4.34}$$

where R is the Ricci scalar of the 2d worldsheet. Before we derive this formula, some quick comments:

- Equation (4.34) holds for any state in the theory — not just the vacuum. This reflects the fact that it comes from regulating short distant divergences in the theory. But, at short distances all finite energy states look basically the same.
- Because $\langle T^{\alpha}_{\ \alpha}\rangle$ is the same for any state it must be equal to something that depends only on the background metric. This something should be local and must be dimension 2. The only candidate is the Ricci scalar R. For this reason, the formula $\langle T^{\alpha}_{\ \alpha}\rangle \sim R$ is the most general possibility. The only question is: what is the coefficient. And, in particular, is it non-zero?

- By a suitable choice of coordinates, we can always put any 2d metric in the form $g_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}$. In these coordinates, the Ricci scalar is given by

$$R = -2e^{-2\omega}\partial^2\omega \tag{4.35}$$

  which depends explicitly on the function ω. Equation (4.34) is then telling us that any conformal theory with $c \neq 0$ has at least one physical observable, $\langle T^{\alpha}_{\ \alpha}\rangle$, which takes different values on backgrounds related by a Weyl transformation ω. This result is referred to as the *Weyl anomaly*, or sometimes as the trace anomaly.

- There is also a Weyl anomaly for conformal field theories in higher dimensions. For example, $4d$ CFTs are characterized by two numbers, a and c, which appear as coefficients in the Weyl anomaly,

$$\langle T^{\mu}_{\ \mu}\rangle_{4d} = \frac{c}{16\pi^2}\, C_{\rho\sigma\kappa\lambda}C^{\rho\sigma\kappa\lambda} - \frac{a}{16\pi^2}\tilde{R}_{\rho\sigma\kappa\lambda}\tilde{R}^{\rho\sigma\kappa\lambda}$$

  where C is the Weyl tensor and $\tilde{R}$ is the dual of the Riemann tensor.

- Equation (4.34) involves only the left-moving central charge c. You might wonder what's special about the left-moving sector. The answer, of course, is nothing. We also have

$$\langle T^{\alpha}_{\ \alpha}\rangle = -\frac{\tilde{c}}{12}\, R$$

  In flat space, conformal field theories with different c and $\tilde{c}$ are perfectly acceptable. However, if we wish these theories to be consistent in fixed, curved backgrounds, then we require $c = \tilde{c}$. This is an example of a *gravitational anomaly*.

- The fact that Weyl invariance requires $c = 0$ will prove crucial in string theory. We shall return to this in Chapter 5.

We will now prove the Weyl anomaly formula (4.34). Firstly, we need to derive an intermediate formula: the $T_{z\bar{z}}\, T_{w\bar{w}}$ OPE. Of course, in the classical theory we found that conformal invariance requires $T_{z\bar{z}} = 0$. We will now show that it's a little more subtle in the quantum theory.

Our starting point is the equation for energy conservation,

$$\partial T_{z\bar{z}} = -\bar{\partial}\, T_{zz}$$

Using this, we can express our desired OPE in terms of the familiar TT OPE,

$$\partial_z T_{z\bar{z}}(z,\bar{z})\ \partial_w T_{w\bar{w}}(w,\bar{w}) = \bar{\partial}_{\bar{z}} T_{zz}(z,\bar{z})\ \bar{\partial}_{\bar{w}} T_{ww}(w,\bar{w}) = \bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\left[\frac{c/2}{(z-w)^4} + \ldots\right] \tag{4.36}$$

Now you might think that the right-hand-side just vanishes: after all, it is an anti-holomorphic derivative $\bar\partial$ of a holomorphic quantity. But we shouldn't be so cavalier because there is a singularity at $z = w$. For example, consider the following equation,

$$\bar\partial_{\bar z}\partial_z \ln|z-w|^2 = \bar\partial_{\bar z}\frac{1}{z-w} = 2\pi\delta(z-w,\bar z-\bar w) \tag{4.37}$$

We proved this statement after equation (4.21). (The factor of 2 difference from (4.21) can be traced to the conventions we defined for complex coordinates in Section 4.0.1). Looking at the intermediate step in (4.37), we again have an anti-holomorphic derivative of a holomorphic function and you might be tempted to say that this also vanishes. But you'd be wrong: subtle things happen because of the singularity and equation (4.37) tells us that the function $1/z$ secretly depends on $\bar z$. (This should really be understood as a statement about distributions, with the delta function integrated against arbitrary test functions). Using this result, we can write

$$\bar\partial_{\bar z}\bar\partial_{\bar w}\frac{1}{(z-w)^4} = \frac{1}{6}\,\bar\partial_{\bar z}\bar\partial_{\bar w}\left(\partial_z^2\,\partial_w\frac{1}{z-w}\right) = \frac{\pi}{3}\,\partial_z^2\,\partial_w\bar\partial_{\bar w}\,\delta(z-w,\bar z-\bar w)$$

Inserting this into the correlation function (4.36) and stripping off the $\partial_z\partial_w$ derivatives on both sides, we end up with what we want,

$$T_{z\bar z}(z,\bar z)\;T_{w\bar w}(w,\bar w) = \frac{c\pi}{6}\,\partial_z\bar\partial_{\bar w}\,\delta(z-w,\bar z-\bar w) \tag{4.38}$$

So the OPE of $T_{z\bar z}$ and $T_{w\bar w}$ almost vanishes, but there's some strange singular behaviour going on as $z\to w$. This is usually referred to as a contact term between operators and, as we have shown, it is needed to ensure the conservation of energy-momentum. We will now see that this contact term is responsible for the Weyl anomaly.

We assume that $\langle T^\alpha{}_\alpha\rangle = 0$ in flat space. Our goal is to derive an expression for $\langle T^\alpha{}_\alpha\rangle$ close to flat space. Firstly, consider the change of $\langle T^\alpha{}_\alpha\rangle$ under a general shift of the metric $\delta g_{\alpha\beta}$. Using the definition of the energy-momentum tensor (4.4), we have

$$\begin{aligned}\delta\langle T^\alpha{}_\alpha(\sigma)\rangle &= \delta\int\mathcal{D}\phi\;e^{-S}\,T^\alpha{}_\alpha(\sigma)\\ &= \frac{1}{4\pi}\int\mathcal{D}\phi\;e^{-S}\left(T^\alpha{}_\alpha(\sigma)\int d^2\sigma'\sqrt{g}\;\delta g^{\beta\gamma}\,T_{\beta\gamma}(\sigma')\right)\end{aligned}$$

If we now restrict to a Weyl transformation, the change to a flat metric is $\delta g_{\alpha\beta} = 2\omega\delta_{\alpha\beta}$, so the change in the inverse metric is $\delta g^{\alpha\beta} = -2\omega\delta^{\alpha\beta}$. This gives

$$\delta\langle T^\alpha{}_\alpha(\sigma)\rangle = -\frac{1}{2\pi}\int\mathcal{D}\phi\;e^{-S}\left(T^\alpha{}_\alpha(\sigma)\int d^2\sigma'\;\omega(\sigma')\,T^\beta{}_\beta(\sigma')\right) \tag{4.39}$$

Now we see why the OPE (4.38) determines the Weyl anomaly. We need to change between complex coordinates and Cartesian coordinates, keeping track of factors of 2. We have

$$T^{\alpha}_{\ \alpha}(\sigma)\, T^{\beta}_{\ \beta}(\sigma') = 16\, T_{z\bar{z}}(z,\bar{z})\; T_{w\bar{w}}(w,\bar{w})$$

Meanwhile, using the conventions laid down in 4.0.1, we have $8\partial_z\bar{\partial}_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}) = -\partial^2\,\delta(\sigma-\sigma')$. This gives us the OPE in Cartesian coordinates

$$T^{\alpha}_{\ \alpha}(\sigma)\, T^{\beta}_{\ \beta}(\sigma') = -\frac{c\pi}{3}\,\partial^2\,\delta(\sigma-\sigma')$$

We now plug this into (4.39) and integrate by parts to move the two derivatives onto the conformal factor ω. We're left with,

$$\delta\,\langle T^{\alpha}_{\ \alpha}\rangle = \frac{c}{6}\,\partial^2\omega \quad \Rightarrow \quad \langle T^{\alpha}_{\ \alpha}\rangle = -\frac{c}{12}R$$

where, to get to the final step, we've used (4.35) and, since we're working infinitesimally, we can replace $e^{-2\omega} \approx 1$. This completes the proof of the Weyl anomaly, at least for spaces infinitesimally close to flat space. The fact that R remains on the right-hand-side for general 2d surfaces follows simply from the comments after equation (4.34), most pertinently the need for the expression to be reparameterization invariant.

4.4.3 c is for Cardy

The Casimir effect and the Weyl anomaly have a similar smell. In both, the central charge provides an extra contribution to the energy. We now demonstrate a different avatar of the central charge: it tells us the density of high energy states.

We will study conformal field theory on a Euclidean torus. We'll keep our normalization $\sigma \in [0, 2\pi)$, but now we also take τ to be periodic, lying in the range

$$\tau \in [0, \beta)$$

The partition function of a theory with periodic Euclidean time has a very natural interpretation: it is related to the free energy of the theory at temperature $T = 1/\beta$.

$$Z[\beta] = \mathrm{Tr}\, e^{-\beta H} = e^{-\beta F} \tag{4.40}$$

At very low temperatures, $\beta \to \infty$, the free energy is dominated by the lowest energy state. All other states are exponentially suppressed. But we saw in 4.4.1 that the vacuum state on the cylinder has Casimir energy $H = -c/12$. In the limit of low temperature, the partition function is therefore approximated by

$$Z \;\to\; e^{c\beta/12} \qquad \text{as } \beta \to \infty \tag{4.41}$$

Now comes the trick. In Euclidean space, both directions of the torus are on equal footing. We're perfectly at liberty to decide that σ is "time" and τ is "space". This can't change the value of the partition function. So let's make the swap. To compare to our original partition function, we want the spatial direction to have range $[0, 2\pi)$. Happily, due to the conformal nature of our theory, we arrange this through the scaling

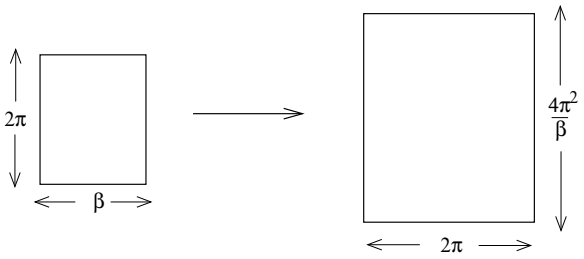


Figure 23:

$$\tau \to \frac{2\pi}{\beta}\,\tau \quad , \quad \sigma \to \frac{2\pi}{\beta}\,\sigma$$

Now we're back where we started, but with the temporal direction taking values in $\sigma \in [0, 4\pi^2/\beta)$. This tells us that the high-temperature and low-temperature partition functions are related,

$$Z[4\pi^2/\beta] = Z[\beta]$$

This is called modular invariance. We'll come across it again in Section 6.4. Writing $\beta' = 4\pi^2/\beta$, this tells us the very high temperature behaviour of the partition function

$$Z[\beta'] \;\to\; e^{c\pi^2/3\beta'} \quad \text{as} \quad \beta' \to 0$$

But the very high temperature limit of the partition function is sampling all states in the theory. On entropic grounds, this sampling is dominated by the high energy states. So this computation is telling us how many high energy states there are.

To see this more explicitly, let's do some elementary manipulations in statistical mechanics. Any system has a density of states $\rho(E) = e^{S(E)}$, where $S(E)$ is the entropy. The free energy is given by

$$e^{-\beta F} = \int dE\; \rho(E)\, e^{-\beta E} = \int dE\; e^{S(E)-\beta E}$$

In two dimensions, all systems have an entropy which scales at large energy as

$$S(E) \to N\sqrt{E} \tag{4.42}$$

The coefficient N counts the number of degrees of freedom. The fact that $S \sim \sqrt{E}$ is equivalent to the fact that $F \sim T^2$, as befits an energy density in a theory with one

spatial dimension. To see this, we need only approximate the integral by the saddle point $S'(E_\star) = \beta$. From (4.42), this gives us the free energy

$$F \sim N^2 T^2$$

We can now make the statement about the central charge more explicit. In a conformal field theory, the entropy of high energy states is given by

$$S(E) \sim \sqrt{cE}$$

This is Cardy's formula.

4.4.4 c has a Theorem

The connection between the central charge and the degrees of freedom in a theory is given further weight by a result of Zamalodchikov, known as the *c-theorem*. The idea of the c-theorem is to stand back and look at the space of all theories and the renormalization group (RG) flows between them.

Conformal field theories are special. They are the fixed points of the renormalization group, looking the same at all length scales. One can consider perturbing a conformal field theory by adding an extra term to the action,

$$S \to S + \alpha \int d^2\sigma \; \mathcal{O}(\sigma)$$

Here $\mathcal{O}$ is a local operator of the theory, while α is some coefficient. These perturbations fall into three classes, depending on the dimension Δ of $\mathcal{O}$.

- $\Delta < 2$: In this case, α has positive dimension: $[\alpha] = 2 - \delta$. Such deformations are called *relevant* because they are important in the infra-red. RG flow takes us away from our original CFT. We only stop flowing when we hit a new CFT (which could be trivial with $c = 0$).

- $\Delta = 2$: The constant α is dimensionless. Such deformations are called *marginal*. The deformed theory defines a new CFT.

- $\Delta > 2$: The constant α has negative dimension. These deformations are irrelevant. The infra-red physics is still described by the original CFT. But the ultra-violet physics is altered.

We expect information is lost as we flow from an ultra-violet theory to the infra-red. The c-theorem makes this intuition precise. The theorem exhibits a function c on the space of all theories which monotonically decreases along RG flows. At the fixed points, c coincides with the central charge of the CFT.

4.5 The Virasoro Algebra

So far our discussion has been limited to the operators of the CFT. We haven't said anything about states. We now remedy this. We start by taking a closer look at the map between the cylinder and the plane.

4.5.1 Radial Quantization

To discuss states in a quantum field theory we need to think about where they live and how they evolve. For example, consider a two dimensional quantum field theory defined on the plane. Traditionally, when quantizing this theory, we parameterize the plane by Cartesian coordinates (t, x) which we'll call "time" and "space". The states live on spatial slices. The Hamiltonian generates time translations and hence governs the evolution of states.

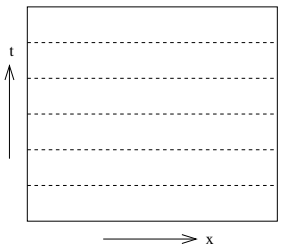


Figure 24:

However, the map between the cylinder and the plane suggests a different way to quantize a CFT on the plane. The complex coordinate on the cylinder is taken to be ω, while the coordinate on the plane is z. They are related by,

$$\omega = \sigma + i\tau \quad , \quad z = e^{-i\omega}$$

On the cylinder, states live on spatial slices of constant σ and evolve by the Hamiltonian,

$$H = \partial_\tau$$

After the map to the plane, the Hamiltonian becomes the dilatation operator

$$D = z\partial + \bar{z}\bar{\partial}$$

If we want the states on the plane to remember their cylindrical roots, they should live on circles of constant radius. Their evolution is governed by the dilatation operator D. This approach to a theory is known as *radial quantization*.

Usually in a quantum field theory, we're interested in time-ordered correlation functions. Time ordering on the cylinder becomes radial ordering on the plane. Operators in correlation functions are ordered so that those inserted at larger radial distance are moved to the left.

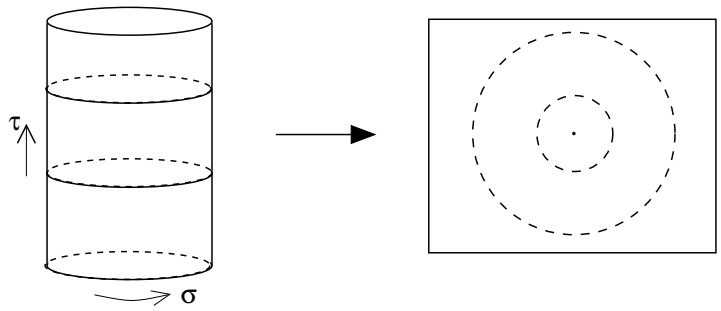


Figure 25: The map from the cylinder to the plane.

Virasoro Generators

Let's look at what becomes of the stress tensor $T(z)$ evaluated on the plane. On the cylinder, we would decompose T in a Fourier expansion.

$$T_{\text{cylinder}}(w) = -\sum_{m=-\infty}^{\infty} L_m e^{imw} + \frac{c}{24}$$

After the transformation (4.33) to the plane, this becomes the Laurent expansion

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

As always, a similar statement holds for the right-moving sector

$$\bar{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

We can invert these expressions to get L_m in terms of $T(z)$. We need to take a suitable contour integral

$$L_n = \frac{1}{2\pi i}\oint dz\; z^{n+1}\, T(z) \qquad , \qquad \tilde{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\; \bar{z}^{n+1}\, \bar{T}(\bar{z}) \tag{4.43}$$

where, if we just want L_n or $\tilde{L}_n$, we must make sure that there are no other insertions inside the contour.

In radial quantization, L_n is the conserved charge associated to the conformal transformation $\delta z = z^{n+1}$. To see this, recall that the corresponding Noether current, given in (4.7), is $J(z) = z^{n+1}T(z)$. Moreover, the contour integral $\oint dz$ maps to the integral around spatial slices on the cylinder. This tells us that L_n is the conserved charge where "conserved" means that it is constant under time evolution on the cylinder, or under radial evolution on the plane. Similarly, $\tilde{L}_n$ is the conserved charge associated to the conformal transformation $\delta\bar{z} = \bar{z}^{n+1}$.

When we go to the quantum theory, conserved charges become generators for the transformation. Thus the operators L_n and $\tilde{L}_n$ generate the conformal transformations $\delta z = z^{n+1}$ and $\delta \bar{z} = \bar{z}^{n+1}$. They are known as the *Virasoro* generators. In particular, our two favorite conformal transformations are

- L_{-1} and $\tilde{L}_{-1}$ generate translations in the plane.
- L_0 and $\tilde{L}_0$ generate scaling and rotations.

The Hamiltonian of the system — which measures the energy of states on the cylinder — is mapped into the dilatation operator on the plane. When acting on states of the theory, this operator is represented as

$$D = L_0 + \tilde{L}_0$$

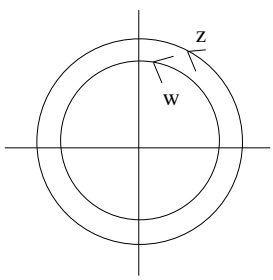
4.5.2 The Virasoro Algebra

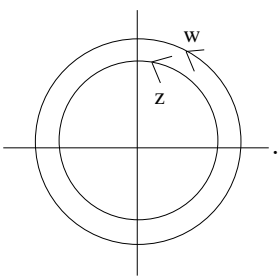
If we have some number of conserved charges, the first thing that we should do is compute their algebra. Representations of this algebra then classify the states of the theory. (For example, think angular momentum in the hydrogen atom). For conformal symmetry, we want to determine the algebra obeyed by the L_n generators. It's a nice fact that the commutation relations are actually encoded TT OPE. Let's see how this works.

We want to compute $[L_m, L_n]$. Let's write L_m as a contour integral over $\oint dz$ and L_n as a contour integral over $\oint dw$. (Note: both z and w denote coordinates on the complex plane now). The commutator is

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \; - \; \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1} w^{n+1} \, T(z) \, T(w)$$

What does this actually mean?! We need to remember that all operator equations are to be viewed as living inside time-ordered correlation functions. Except, now we're working on the z-plane, this statement has transmuted into radially ordered correlation functions: outies to the left, innies to the right.

So $L_m L_n$ means 

while $L_n L_m$ means 

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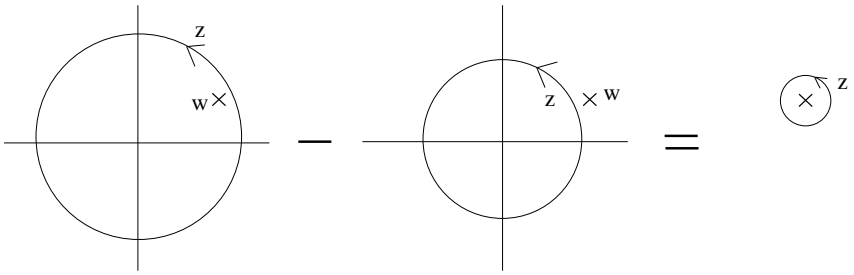
The trick to computing the commutator is to first fix w and do the $\oint dz$ integrations. The resulting contour is,



In other words, we do the z-integration around a fixed point w, to get

$$\begin{aligned}[L_m, L_n] &= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i}\; z^{m+1} w^{n+1}\, T(z)\, T(w) \\ &= \oint \frac{dw}{2\pi i}\, \text{Res}\left[z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \right) \right]\end{aligned}$$

To compute the residue at $z = w$, we first need to Taylor expand z^{m+1} about the point w,

$$\begin{aligned}z^{m+1} = w^{m+1} + (m+1)w^m(z-w) + \frac{1}{2}m(m+1)w^{m-1}(z-w)^2 \\ + \frac{1}{6}m(m^2-1)w^{m-2}(z-w)^3 + \ldots\end{aligned}$$

The residue then picks up a contribution from each of the three terms,

$$[L_m, L_n] = \oint \frac{dw}{2\pi i}\; w^{n+1} \left[w^{m+1} \partial T(w) + 2(m+1) w^m\, T(w) + \frac{c}{12} m(m^2-1) w^{m-2} \right]$$

To proceed, it is simplest to integrate the first term by parts. Then we do the w-integral. But for both the first two terms, the resulting integral is of the form (4.43) and gives us L_{m+n}. For the third term, we pick up the pole. The end result is

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

This is the *Virasoro algebra*. It's quite famous. The $\tilde{L}_n$'s satisfy exactly the same algebra, but with c replaced by $\tilde{c}$. Of course, $[L_n, \tilde{L}_m] = 0$. The appearance of c as an extra term in the Virasoro algebra is the reason it is called the "central charge". In general, a central charge is an extra term in an algebra that commutes with everything else.

Conformal = Diffeo + Weyl

We can build some intuition for the Virasoro algebra. We know that the L_n's generate conformal transformations $\delta z = z^{n+1}$. Let's consider something closely related: a coordinate transformation $\delta z = z^{n+1}$. These are generated by the vector fields

$$l_n = z^{n+1}\partial_z \tag{4.44}$$

But it's a simple matter to compute their commutation relations:

$$[l_n, l_m] = (m-n)l_{m+n}$$

So this is giving us the first part of the Virasoro algebra. But what about the central term? The key point to remember is that, as we stressed at the beginning of this chapter, a conformal transformation is not just a reparameterization of the coordinates: it is a reparameterization, followed by a compensating Weyl rescaling. The central term in the Virasoro algebra is due to the Weyl rescaling.

4.5.3 Representations of the Virasoro Algebra

With the algebra of conserved charges at hand, we can now start to see how the conformal symmetry classifies the states into representations.

Suppose that we have some state $|\psi\rangle$ that is an eigenstate of L_0 and $\tilde{L}_0$.

$$L_0 |\psi\rangle = h |\psi\rangle \quad , \quad \tilde{L}_0 |\psi\rangle = \tilde{h} |\psi\rangle$$

Back on the cylinder, this corresponds to some state with energy

$$\frac{E}{2\pi} = h + \tilde{h} - \frac{c+\tilde{c}}{24}$$

For this reason, we'll refer to the eigenvalues h and $\tilde{h}$ as the energy of the state. By acting with the L_n operators, we can get further states with eigenvalues

$$L_0 L_n |\psi\rangle = (L_n L_0 - nL_n) |\psi\rangle = (h-n)L_n |\psi\rangle$$

This tells us that L_n are raising and lowering operators depending on the sign of n. When $n > 0$, L_n lowers the energy of the state and L_{-n} raises the energy of the state. If the spectrum is to be bounded below, there must be some states which are annihilated by all L_n and $\tilde{L}_n$ for $n > 0$. Such states are called *primary*. They obey

$$L_n |\psi\rangle = \tilde{L}_n |\psi\rangle = 0 \quad \text{for all } n > 0$$

In the language of representation theory, they are also called highest weight states. They are the states of lowest energy.

Representations of the Virasoro algebra can now be built by acting on the primary states with raising operators L_{-n} with $n > 0$. Obviously this results in an infinite tower of states. All states obtained in this way are called *descendants*. From an initial primary state $|\psi\rangle$, the tower fans out...

$$\begin{gathered} |\psi\rangle \\ L_{-1}|\psi\rangle \\ L_{-1}^2|\psi\rangle \ , \ L_{-2}|\psi\rangle \\ L_{-1}^3|\psi\rangle \ , \ L_{-1}L_{-2}|\psi\rangle \ , \ L_{-3}|\psi\rangle \end{gathered}$$

The whole set of states is called a *Verma* module. They are the irreducible representations of the Virasoro algebra. This means that if we know the spectrum of primary states, then we know the spectrum of the whole theory.

Some comments:

- The vacuum state $|0\rangle$ has $h = 0$. This state obeys

$$L_n|0\rangle = 0 \quad \text{for all } n \geq 0 \tag{4.45}$$

  Note that this state preserves the maximum number of symmetries: like all primary states, it is annihilated by L_n with $n > 0$, but it is also annihilated by L_0. This fits with our intuition that the vacuum state should be invariant under as many symmetries as possible. You might think that we could go further and require that the vacuum state obeys $L_n|0\rangle = 0$ for all n. But that isn't consistent with the central charge term in Virasoro algebra. The requirements (4.45) are the best we can do.

- This discussion should be ringing bells. We saw something very similar in the covariant quantization of the string, where we imposed conditions (2.6) as constraints. We will see the connection between the primary states and the spectrum of the string in Section 5.

- There's a subtlety that you should be aware of: the states in the Verma module are not necessarily all independent. It could be that some linear combination of the states vanishes. This linear combination is known as a null state. The existence of null states depends on the values of h and c. For example, suppose that we are in a theory in which the central charge is $c = 2h(5-8h)/(2h+1)$, where h is the energy of a primary state $|\psi\rangle$. Then it is simple to check that the following combination has vanishing norm:

$$L_{-2}|\psi\rangle - \frac{3}{2(2h+1)}L_{-1}^2|\psi\rangle \tag{4.46}$$

- There is a close relationship between the primary states and the primary operators defined in Section 4.2.3. In fact, the energies h and $\tilde{h}$ of primary states will turn out to be exactly the weights of primary operators in the theory. This connection will be described in Section 4.6.

4.5.4 Consequences of Unitarity

There is one physical requirement that a theory must obey which we have so far neglected to mention: *unitarity*. This is the statement that probabilities are conserved when we are in Minkowski signature spacetime. Unitarity follows immediately if we have a Hermitian Hamiltonian which governs time evolution. But so far our discussion has been somewhat algebraic and we've not enforced this condition. Let's do so now.

We retrace our footsteps back to the Euclidean cylinder and then back again to the Minkowski cylinder where we can ask questions about time evolution. Here the Hamiltonian density takes the form

$$\mathcal{H} = T_{ww} + T_{\bar{w}\bar{w}} = \sum_n L_n e^{-in\sigma^+} + \tilde{L}_n e^{-in\sigma^-}$$

So for the Hamiltonian to be Hermitian, we require

$$L_n = L_{-n}^\dagger$$

This requirement imposes some strong constraints on the structure of CFTs. Here we look at a couple of trivial, but important, constraints that arise due to unitarity and the requirement that the physical Hilbert space does not contain negative norm states.

- $h \geq 0$: This fact follows from looking at the norm,

  $$|L_{-1}|\psi\rangle|^2 = \langle\psi| L_{+1}L_{-1}|\psi\rangle = \langle\psi|[L_{+1}, L_{-1}]|\psi\rangle = 2h\langle\psi|\psi\rangle \geq 0$$

  The only state with $h = 0$ is the vacuum state $|0\rangle$.

- $c > 0$: To see this, we can look at

  $$|L_{-n}|0\rangle|^2 = \langle 0|[L_n, L_{-n}]|0\rangle = \frac{c}{12}n(n^2-1) \geq 0 \tag{4.47}$$

  So $c \geq 0$. If $c = 0$, the only state in the vacuum module is the vacuum itself. It turns out that, in fact, the only state in the whole theory is the vacuum itself. Any non-trivial CFT has $c > 0$.

There are many more requirements of this kind that constrain the theory. In fact, it turns out that for CFTs with $c < 1$ these requirements are enough to classify and solve all theories.

4.6 The State-Operator Map

In this section we describe one particularly important aspect of conformal field theories: a map between states and local operators.

Firstly, let's get some perspective. In a typical quantum field theory, the states and local operators are very different objects. While local operators live at a point in spacetime, the states live over an entire spatial slice. This is most clear if we write down a Schrödinger-style wavefunction. In field theory, this object is actually a wavefunctional, $\Psi[\phi(\sigma)]$, describing the probability for every field configuration $\phi(\sigma)$ at each point σ in space (but at a fixed time).

Given that states and local operators are such very different beasts, it's a little surprising that in a CFT there is an isomorphism between them: it's called the state-operator map. The key point is that the distant past in the cylinder gets mapped to a single point $z = 0$ in the complex plane. So specifying a state on the cylinder in the far past is equivalent to specifying a local disturbance at the origin.

To make this precise, we need to recall how to write down wavefunctions using path integrals. Different states are computed by putting different boundary conditions on the functional integral. Let's start by returning to quantum mechanics and reviewing a few simple facts. The propagator for a particle to move from position x_i at time τ_i to position x_f at time τ_f is given by

$$G(x_f, x_i) = \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x \; e^{iS}$$

This means that if our system starts off in some state described by the wavefunction $\psi_i(x_i)$ at time τ_i then (ignoring the overall normalization) it evolves to the state

$$\psi_f(x_f, \tau_f) = \int dx_i \, G(x_f, x_i) \, \psi_i(x_i, \tau_i)$$

There are two lessons to take from this. Firstly, to determine the value of the wavefunction at a given point x_f, we evaluate the path integral restricting to paths which satisfy $x(\tau_f) = x_f$. Secondly, the initial state $\psi(x_i)$ acts as a weighting factor for the integral over initial boundary conditions.

Let's now write down the same formula in a field theory, where we're dealing with wavefunctionals. We'll work with the Euclidean path integral on the cylinder. If we start with some state $\Psi_i[\phi_i(\sigma)]$ at time τ_i, then it will evolve to the state

$$\Psi_f[\phi_f(\sigma), \tau_f] = \int \mathcal{D}\phi_i \int_{\phi(\tau_i)=\phi_i}^{\phi(\tau_f)=\phi_f} \mathcal{D}\phi \; e^{-S[\phi]} \; \Psi_i[\phi_i(\sigma), \tau_i]$$

How do we write a similar expression for states after the map to the complex plane? Now the states are defined on circles of constant radius, say $|z| = r$, and evolution is governed by the dilatation operator. Suppose the initial state is defined at $|z| = r_i$. In the path integral, we integrate over all fields with fixed boundary conditions $\phi(r_i) = \phi_i$ and $\phi(r_f) = \phi_f$ on the two edges of the annulus shown in the figure,

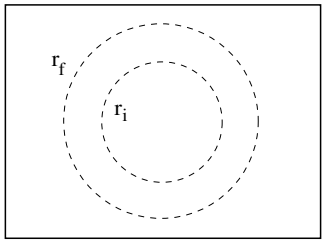


Figure 26:

$$\Psi_f[\phi_f(\sigma), r_f] = \int \mathcal{D}\phi_i \int_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} \mathcal{D}\phi \; e^{-S[\phi]} \; \Psi_i[\phi_i(\sigma), r_i]$$

This is the traditional way to define a state in field theory, albeit with a slight twist because we're working in radial quantization. We see that the effect of the initial state is to change the weighting of the path integral over the inner ring at $|z| = r_i$.

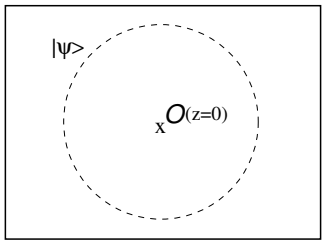


Figure 27:

Let's now see what happens as we take the initial state back to the far past and, ultimately, to $z = 0$? We must now integrate over the whole disc $|z| \leq r_f$, rather than the annulus. The only effect of the initial state is now to change the weighting of the path integral at the point $z = 0$. But that's exactly what we mean by a local operator inserted at that point. This means that each local operator $\mathcal{O}(z = 0)$ defines a different state in the theory,

$$\Psi[\phi_f; r] = \int^{\phi(r)=\phi_f} \mathcal{D}\phi \; e^{-S[\phi]} \, \mathcal{O}(z = 0)$$

We're now integrating over all field configurations within the disc, including all possible values of the field at $z = 0$, which is analogous to integrating over the boundary conditions $\int \mathcal{D}\phi_i$ on the inner circle.

- The state-operator map is only true in conformal field theories where we can map the cylinder to the plane. It also holds in conformal field theories in higher dimensions (where $\mathbf{R} \times \mathbf{S}^{D-1}$ can be mapped to the plane $\mathbf{R}^D$). In non-conformal field theories, a typical local operator creates many different states.

- The state-operator map does not say that the number of states in the theory is equal to the number of operators: this is never true. It does say that the states are in one-to-one correspondence with the *local* operators.

- You might think that you've seen something like this before. In the canonical quantization of free fields, we create states in a Fock space by acting with creation operators. That's *not* what's going on here! The creation operators are just about as far from local operators as you can get. They are the Fourier transforms of local operators.

- There's a special state that we can create this way: the vacuum. This arises by inserting the identity operator $\mathbf{1}$ into the path integral. Back in the cylinder picture, this just means that we propagate the state back to time $\tau = -\infty$ which is a standard trick used in the Euclidean path integral to project out all but the ground state. For this reason the vacuum is sometimes referred to, in operator notation, as $|\mathbf{1}\rangle$.

4.6.1 Some Simple Consequences

Let's use the state-operator map to wrap up a few loose ends that have arisen in our study of conformal field theory.

Firstly, we've defined two objects that we've called "primary": states and operators. The state-operator map relates the two. Consider the state $|\mathcal{O}\rangle$, built from inserting a primary operator $\mathcal{O}$ into the path integral at $z = 0$. We can look at,

$$\begin{aligned} L_n |\mathcal{O}\rangle &= \oint \frac{dz}{2\pi i}\, z^{n+1}\, T(z)\, \mathcal{O}(z=0) \\ &= \oint \frac{dz}{2\pi i}\, z^{n+1} \left(\frac{h\mathcal{O}}{z^2} + \frac{\partial \mathcal{O}}{z} + \ldots \right) \end{aligned} \tag{4.48}$$

You may wonder what became of the path integral $\int \mathcal{D}\phi\, e^{-S[\phi]}$ in this expression. The answer is that it's still implicitly there. Remember that operator expressions such as (4.43) are always taken to hold inside correlation functions. But putting an operator in the correlation function is the same thing as putting it in the path integral, weighted with $e^{-S[\phi]}$.

From (4.48) we can see the effect of various generators on states

- $L_{-1} |\mathcal{O}\rangle = |\partial \mathcal{O}\rangle$: In fact, this is true for all operators, not just primary ones. It is expected since L_{-1} is the translation generator.

- $L_0 |\mathcal{O}\rangle = h |\mathcal{O}\rangle$: This is true of any operator with well defined transformation under scaling.

- $L_n |\mathcal{O}\rangle = 0$ for all $n > 0$. This is true only of primary operators $\mathcal{O}$. Moreover, it is our requirement for $|\mathcal{O}\rangle$ to be a primary state.

This has an important consequence. We stated earlier that one of the most important things to compute in a CFT is the spectrum of weights of primary operators. This seems like a slightly obscure thing to do. But now we see that it has a much more direct, physical meaning. It is the spectrum of energy and angular momentum of states of the theory defined on the cylinder.

Another loose end: when defining operators which carry specific weight, we made the statement that we could always work in a basis of operators which have specified eigenvalues under D and L. This follows immediately from the statement that we can always find a basis of eigenstates of H and L on the cylinder.

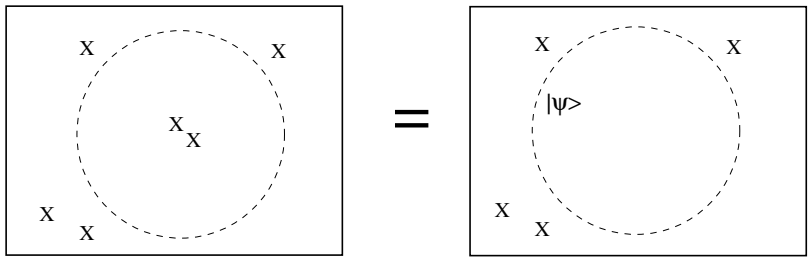


Figure 28:

Finally, we can use this idea of the state-operator map to understand why the OPE works so well in conformal field theories. Suppose that we're interested in some correlation function, with operator insertions as shown in the figure. The statement of the OPE is that we can replace the two inner operators by a sum of operators at $z = 0$, *independent* of what's going on outside of the dotted line. As an operator statement, that sounds rather surprising. But this follows by computing the path integral up to the dotted line, by which point the only effect of the two operators is to determine what state we have. This provides us a way of understanding why the OPE is exact in CFTs, with a radius of convergence equal to the next-nearest insertion.

4.6.2 Our Favourite Example: The Free Scalar Field

Let's illustrate the state-operator map by returning yet again to the free scalar field. On a Euclidean cylinder, we have the mode expansion

$$X(w,\bar{w}) = x + \alpha' p\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\left(\alpha_n\, e^{inw} + \tilde{\alpha}_n\, e^{in\bar{w}}\right)$$

where we retain the requirement of reality in Minkowski space, which gave us $\alpha_n^\star = \alpha_{-n}$ and $\tilde{\alpha}_n^\star = \tilde{\alpha}_{-n}$. We saw in Section 4.3 that X does not have good conformal properties.

Before transforming to the $z = e^{-iw}$ plane, we should work with the primary field on the cylinder,

$$\partial_w X(w, \bar{w}) = -\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n \, e^{inw} \qquad \text{with } \alpha_0 \equiv i\sqrt{\frac{\alpha'}{2}} p$$

Since ∂X is a primary field of weight $h = 1$, its transformation to the plane is given by (4.18) and reads

$$\partial_z X(z) = \left(\frac{\partial z}{\partial w}\right)^{-1} \partial_w X(w) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\alpha_n}{z^{n+1}}$$

and similar for $\bar{\partial} X$. Inverting this gives an equation for α_n as a contour integral,

$$\alpha_n = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i} \, z^n \, \partial X(z) \tag{4.49}$$

Just as the TT OPE allowed us to determine the $[L_m, L_n]$ commutation relations in the previous section, so the $\partial X \partial X$ OPE contains the information about the $[\alpha_m, \alpha_n]$ commutation relations. The calculation is straightforward,

$$\begin{aligned}
[\alpha_m, \alpha_n] &= -\frac{2}{\alpha'} \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \; - \; \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) \, z^m w^n \, \partial X(z) \, \partial X(w) \\
&= -\frac{2}{\alpha'} \oint \frac{dw}{2\pi i} \, \text{Res}_{z=w} \left[z^m w^n \left(\frac{-\alpha'/2}{(z-w)^2} + \ldots \right) \right] \\
&= m \oint \frac{dw}{2\pi i} \, w^{m+n-1} = m\delta_{m+n,0}
\end{aligned}$$

where, in going from the second to third line, we have Taylor expanded z around w. Hearteningly, the final result agrees with the commutation relation (2.2) that we derived in string theory using canonical quantization.

The State-Operator Map for the Free Scalar Field

Let's now look at the map between states and local operators. We know from canonical quantization that the Fock space is defined by acting with creation operators α_{-m} with $m > 0$ on the vacuum $|0\rangle$. The vacuum state itself obeys $\alpha_m |0\rangle = 0$ for $m > 0$. Finally, there is also the zero mode $\alpha_0 \sim p$ which provides all states with another quantum number. A general state is given by

$$\prod_{m=1}^{\infty} \alpha_{-m}^{k_m} \, |0; p\rangle$$

Let's try and recover these states by inserting operators into the path integral. Our first task is to check whether the vacuum state is indeed equivalent to the insertion of the identity operator. In other words, is the ground state wavefunctional of the theory on the circle $|z| = r$ really given by

$$\Psi_0[X_f] = \int^{X_f(r)} \mathcal{D}X \; e^{-S[X]} \qquad ? \tag{4.50}$$

We want to check that this satisfies the definition of the vacuum state, namely $\alpha_m|0\rangle = 0$ for $m > 0$. How do we act on the wavefunctional with an operator? We should still integrate over all field configurations $X(z, \bar{z})$, subject to the boundary conditions at $X(|z| = r) = X_f$. But now we should insert the contour integral (4.49) at some $|w| < r$ (because, after all, the state is only going to vanish after we've hit it with α_m, not before!). So we look at

$$\alpha_m \Psi_0[X_f] = \int^{X_f} \mathcal{D}X \; e^{-S[X]} \oint \frac{dw}{2\pi i} \, w^m \partial X(w)$$

The path integral is weighted by the action (4.19) for a free scalar field. If a given configuration diverges somewhere inside the disc $|z| < r$, then the action also diverges. This ensures that only smooth functions $\partial X(z)$, which have no singularity inside the disc, contribute. But for such functions we have

$$\oint \frac{dw}{2\pi i} \, w^m \partial X(w) = 0 \qquad \text{for all } m \geq 0$$

So the state (4.50) is indeed the vacuum state. In fact, since α_0 also annihilates this state, it is identified as the vacuum state with vanishing momentum.

What about the excited states of the theory?

Claim: $\alpha_{-m}|0\rangle = |\partial^m X\rangle$. By which we mean that the state $\alpha_{-m}|0\rangle$ can be built from the path integral,

$$\alpha_{-m}|0\rangle = \int \mathcal{D}X \; e^{-S[X]} \; \partial^m X(z = 0) \tag{4.51}$$

Proof: We can check this by acting on $|\partial^m X\rangle$ with the annihilation operators α_n.

$$\alpha_n \, |\partial^m X\rangle \sim \int^{X_f(r)} \mathcal{D}X \; e^{-S[X]} \; \oint \frac{dw}{2\pi i} \; w^n \, \partial X(w) \, \partial^m X(z = 0)$$

We can focus on the operator insertions and use the OPE (4.23). We drop the path integral and just focus on the operator equation (because, after all, operator equations only make sense in correlation functions which is the same thing as in path integrals). We have

$$\oint \frac{dw}{2\pi i}\ w^n\, \partial_z^{m-1} \frac{1}{(w-z)^2}\bigg|_{z=0} = m! \oint \frac{dw}{2\pi i}\ w^{n-m-1} = 0 \qquad \text{unless } m = n$$

This confirms that the state (4.51) has the right properties. □

Finally, we should worry about the zero mode, or momentum $\alpha_0 \sim p$. It is simple to show using the techniques above (together with the OPE (4.26)) that the momentum of a state arises by the insertion of the primary operator e^{ipX}. For example,

$$|0; p\rangle \sim \int \mathcal{D}X\ e^{-S[X]}\ e^{ipX(z=0)}\ .$$

4.7 Brief Comments on Conformal Field Theories with Boundaries

The open string lives on the infinite strip with spatial coordinate $\sigma \in [0, \pi]$. Here we make just a few brief comments on the corresponding conformal field theories.

As before, we can define the complex coordinate $w = \sigma + i\tau$ and make the conformal map

$$z = e^{-iw}$$

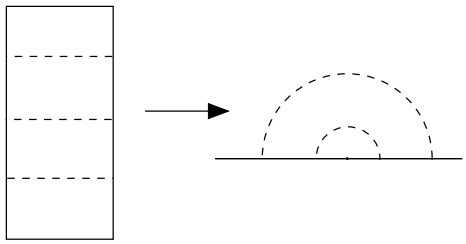


Figure 29:

This time the map takes us to the upper-half plane: $\text{Im}z \geq 0$. The end points of the string are mapped to the real axis, $\text{Im}z = 0$.

Much of our previous discussion goes through as before. But now we need to take care of boundary conditions at $\text{Im}z = 0$. Let's first look at $T_{\alpha\beta}$. Recall that the stress-energy tensor exists because of translational invariance. We still have translational invariance in the direction parallel to the boundary — let's call the associated tangent vector t^α. But translational invariance is broken perpendicular to the boundary — we call the normal vector n^α. The upshot of this is that $T_{\alpha\beta}t^\beta$ remains a conserved current.

To implement Neumann boundary conditions, we insist that none of the current flows out of the boundary. The condition is

$$T_{\alpha\beta}n^\alpha t^\beta = 0 \qquad \text{at } \text{Im}z = 0$$

In complex coordinates, this becomes

$$T_{zz} = T_{\bar{z}\bar{z}} \qquad \text{at } \mathrm{Im} z = 0$$

There's a simple way to implement this: we extend the definition of T_{zz} from the upper-half plane to the whole complex plane by defining

$$T_{zz}(z) = T_{\bar{z}\bar{z}}(\bar{z})$$

For the closed string we had both functions T and $\bar{T}$ in the whole plane. But for the open string, we have just one of these – say, T, — in the whole plane. This contains the same information as both T and $\bar{T}$ in the upper-half plane. It's simpler to work in the whole plane and focus just on T. Correspondingly, we now have just a single set of Virasoro generators,

$$L_n = \oint \frac{dz}{2\pi i}\, z^{n+1}\, T_{zz}(z)$$

There is no independent $\tilde{L}_n$ for the open string.

A similar doubling trick works when computing the propagator for the free scalar field. The scalar field $X(z,\bar{z})$ is only defined in the upper-half plane. Suppose we want to implement Neumann boundary conditions. Then the propagator is defined by

$$\langle X(z,\bar{z})\, X(w,\bar{w})\rangle = G(z,\bar{z};w,\bar{w})$$

which obeys $\partial^2 G = -2\pi\alpha'\,\delta(z-w,\bar{z}-\bar{w})$ subject to the boundary condition

$$\partial_\sigma\, G(z,\bar{z};w,\bar{w})|_{\sigma=0} = 0$$

But we solve problems like this in our electrodynamics courses. A useful way of proceeding is to introduce an "image charge" in the lower-half plane. We now let $X(z,\bar{z})$ vary over the whole complex plane with its dynamics governed by the propagator

$$G(z,\bar{z};w,\bar{w}) = -\frac{\alpha'}{2}\,\ln|z-w|^2 - \frac{\alpha'}{2}\,\ln|z-\bar{w}|^2 \tag{4.52}$$

Much of the remaining discussion of CFTs carries forward with only minor differences. However, there is one point that is simple but worth stressing because it will be of importance later. This concerns the state-operator map. Recall the logic that leads us to this idea: we consider a state at fixed time on the strip and propagate it back to past infinity $\tau \to -\infty$. After the map to the half-plane, past infinity is again the origin. But now the origin lies on the boundary. We learn that the state-operator map relates states to local operators defined on the boundary.

This fact ensures that theories on a strip have fewer states than those on the cylinder. For example, for a free scalar field, Neumann boundary conditions require $\partial X = \bar{\partial} X$ at $\mathrm{Im} z = 0$. (This follows from the requirement that $\partial_\sigma X = 0$ at $\sigma = 0, \pi$ on the strip). On the cylinder, the operators ∂X and $\bar{\partial} X$ give rise to different states; on the strip they give rise to the same state. This, of course, mirrors what we've seen for the quantization of the open string where boundary conditions mean that we have only half the oscillator modes to play with.