Supersymmetry: Example Sheet 2

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1. Determine the (possibly spurious) symmetries of the superpotential

\[ W = \mu_2 \Phi^2 + \mu_3 \Phi^3 + \ldots + \mu_n \Phi^n \]

Argue that the superpotential is not renormalised at any order in perturbation theory.

2*. Find the vacuum structure, including the vacuum energy, of the following theories:

a) A chiral multiplet \( Z \) with \( W = \alpha Z + \beta /Z \), with \( \alpha, \beta \neq 0 \).

b) Three chiral multiplets \( X, Y \) and \( Z \) with \( W = XYZ \).

c) Three chiral multiplets \( X, Y \) and \( Z \) with \( W = \alpha Y + \beta Y X^2 + \gamma XZ \) with \( \alpha, \beta, \gamma \neq 0 \) and \(|\gamma|^2 > 2|\alpha\beta|\).

(You may assume a canonical Kähler potential for all fields.)

3a. For an abelian vector superfield \( V \), show that the field strength superfield

\[ W_\alpha = -\frac{1}{4} \overline{D}^2 D_\alpha V \]

is chiral, i.e. \( \overline{D}_\alpha W_\alpha = 0 \). Show further that it is invariant under extended gauge transformations \( V \rightarrow V + i(\Omega - \Omega^\dagger) \).

b. Show that the components of \( W_\alpha \) are

\[ W_\alpha(x, \theta) = \lambda_\alpha(x) + \theta_\alpha D(x) + (\sigma^{\mu\nu}\theta_\alpha) F_{\mu\nu}(x) - i\theta^2 \sigma^{\mu}_{\alpha\beta} \partial_\mu \bar{\lambda}_\beta(x) + \ldots \]

Hint: you will be well served to work in Wess-Zumino gauge and, at an appropriate time, to use the fact that \( W_\alpha \) is a chiral superfield.

c. Show that the F-term integral gives

\[ \int d^2 \theta W^\alpha W_\alpha = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} F_{\mu\nu}^* F^{\mu\nu} - 2i \lambda \partial_\mu \bar{\lambda} + D^2 \]

Note: You will need the identity

\[ \text{tr}(\sigma^{\mu}_{\alpha\beta} \sigma^{\nu}_{\gamma\delta} \sigma^{\sigma}_{\rho}) = 2i \epsilon^{\mu\nu\sigma\rho} + 2\eta^{\mu\nu} \eta^{\sigma\rho} - 2\eta^{\mu\sigma} \eta^{\nu\rho} + 2\eta^{\mu\rho} \eta^{\nu\sigma} \]
4*. Super Yang-Mills (with vanishing theta angle) has action

\[ S = \frac{1}{g^2} \int d^4x \, \text{Tr} \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i \lambda \sigma^\mu D_\mu \bar{\lambda} \right] \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \) and \( \lambda \) is an adjoint Weyl fermion with \( D_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda] \). Show that the action is invariant under the supersymmetry transformations

\[ \delta A_\mu = \epsilon \sigma^\mu \bar{\lambda} + \lambda \sigma_\mu \bar{\epsilon} \quad \text{and} \quad \delta \lambda = (\sigma^\mu \epsilon) F_{\mu\nu} \]

Hint: You can check the supersymmetry transformation just for \( \epsilon \) with the \( \bar{\epsilon} \) terms guaranteed to follow suit on the grounds that the action is real. To do the calculation, you will need to invoke the Bianchi identity \( D_\mu \star F_{\mu\nu} = 0 \) together with the sigma-matrix identity

\[ \sigma^\nu \sigma^\mu \sigma^\rho = \eta^\mu\nu \sigma^\rho + \eta^\mu\rho \sigma^\nu - \eta^\nu\rho \sigma^\mu + i \epsilon^{\nu\mu\rho\kappa} \sigma_\kappa \]

5. A \( U(N_c) \) supersymmetric gauge theory is coupled to \( N_f \) flavours, comprising of chiral multiplets \( \Phi_i \) in the fundamental representation and \( \tilde{\Phi}_i \) in the anti-fundamental with \( i = 1, \ldots, N_f \). The D-terms conditions are

\[ D^A = \phi_i^A T^A \phi^i - \bar{\phi}_i T^A \bar{\phi}^i = 0 \quad A = 1, \ldots, N_c^2 \]

By constructing an explicit set of Hermitian generators \( (T^A)_a^b \), with \( a, b = 1, \ldots, N_c \), show that the conditions \( D^A = 0 \) are equivalent to

\[ \phi_i^a \phi_b^i - \bar{\phi}_i \bar{\phi}_b^i = 0 \quad a, b = 1, \ldots, N_c \]

6. An \( SU(2) \) gauge theory is coupled to three chiral multiplets, each in the adjoint representation. The theory has a superpotential given by

\[ W = \text{Tr} \left( \Phi_1 [\Phi_2, \Phi_3] - \frac{m}{2} \sum_{i=1}^3 \Phi_i^2 \right) \]

Write down the \( D \)-term and \( F \)-term contributions to the potential energy. Show that the zero energy ground states obey

\[ [\phi_i, \phi_j] = m \epsilon_{ijk} \phi_k \quad \text{and} \quad \sum_{i=1}^3 [\phi_i, \phi_i^+] = 0 \]

What are the solutions to these equations? What is the surviving gauge symmetry in each ground state?