Supersymmetric Field Theory

University of Cambridge Part III Mathematical Tripos

David Tong

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge, CB3 OBA, UK

http://www.damtp.cam.ac.uk/user/tong/susy.html
d.tong@damtp.cam.ac.uk

Recommended Books and Resources

Here is a collection of useful textbooks on supersymmetry.

- Wess and Bagger "*Supersymmetry*"

This is a strange little book, with chapters that are 2 pages long followed by several pages of key equations. It's not particularly good for learning the subject, but makes a remarkably useful reference guide.

- Bailin and Love "*Supersymmetric Gauge Field Theory and String Theory*"

Probably the best book covering the basics of supersymmetric Lagrangians.

- Dan Freedman and Toine Van Proeyen "*Supergravity*"

As the name suggests, this book is mostly focussed on supergravity rather than global supersymmetry. But it kicks off with a really excellent description of classical field theory. The section on spinors in various dimensions is particularly useful.

- Steven Weinberg "*The Quantum Theory of Fields, Volume III: Supersymmetry*"

The third volume of Weinberg's magnum opus covers supersymmetry. As always, it contains many important things that are difficult to find elsewhere. As always, these things are sometimes frustratingly buried in unconventional notation and dressed with more indices than you can shake a stick at.

- John Terning "*Modern Supersymmetry: Dynamics and Duality*"

This is one of the few books (possibly the only book) that describes the quantum dynamics of supersymmetric field theories, rather than just their classical action. (Weinberg has a chapter on the Seiberg-Witten solution, but it feels like his heart isn't in it and any mention of Seiberg duality is noticeably absent.) There are, fortunately, many lecture notes that make up for the deficiency. You can find links on the course webpage.

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Acknowledgements

These lecture notes owe a debt to Graham Shore, from whom I first learned supersymmetry, and to Philip Argyres' wonderful notes on the subject. I'm also grateful to Ben Allanach and Fernando Quevedo, both previous lecturers of the supersymmetry course in Cambridge.

This is one of the more advanced courses in Part III. It assumes a familiarity with quantum field theory, in particular the renormalisation group. You will also need to be comfortable with some group theory.

Spinor Conventions

We work in Minkowski space with signature $(+,-,-,-)$. Spinor indices are raised and lowered with $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$ where the invariant, anti-symmetric tensor is

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Left-handed spinors are contracted as $\psi\chi = \psi^\alpha\chi_\alpha$ and right-handed spinors are contracted as $\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$. Sigma matrices are defined by

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \sigma^i)_{\alpha\dot{\alpha}} \quad \text{and} \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} = (1, -\sigma^i)^{\dot{\alpha}\alpha}$$

and the generators of the Lorentz group in the left-handed and right-handed spinor representation are, respectively,

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta} \quad \text{and} \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} = \frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}}$$

1 Introduction

Supersymmetry is the name given to a novel symmetry that relates bosons and fermions. In many ways it is a surprise that such a symmetry could exist at all. This is because bosons and fermions are, to put it mildly, different.

Bosons are gregarious. Put many of them in a box and they huddle together to form a macroscopic quantum object called a Bose-Einstein condensate. In contrast, fermions are loners, an isolation enforced by the Pauli exclusion principle. Put many fermions in a box and you get a more familiar, but ultimately even stranger, state of quantum matter called a Fermi surface.

Within the framework of relativistic quantum field theories, the difference between fermions and bosons is even more stark. Fermions are matter particles. Bosons are force carriers. Any symmetry that relates the two must somehow entail a unification of matter and force.

Of course, we know from our earlier lessons on Quantum Field Theory that the distinction between bosons and fermion can be traced to something that is, in some sense, rather minor. They differ only by the simple matter of $\hbar/2$ in their angular momentum, with the spin-statistics theorem then doing the heavy lifting that ensures the resulting particles have such different properties. However, this too highlights just how unusual supersymmetry must be. The angular momentum of a particle is a property that follows from the symmetries of spacetime. Anything that relates particles with different angular momentum must involve some kind of extension of the symmetries of spacetime. And that sounds interesting!

All of this means that it's not at all obvious that something like supersymmetry can exist and we should, if nothing else, be curious about how it can come about. But why else should we care? In the rest of this introduction, I give three reasons why studying supersymmetric quantum field theories is worthwhile.

Reason 1: Strongly Interacting Quantum Field Theories

Quantum field theory is hard. This is particularly true when coupling constants, which specify the strength of interactions, are not small. This means that we can no longer understand the physics using the familiar methods of perturbation theory and Feynman diagrams. In this case, the word "hard" typically means "no one knows how to solve it".

Supersymmetric theories are not wildly different from other quantum field theories. They have a carefully curated collection of fields, with some interactions tuned to take certain values, but otherwise they exhibit many of the strongly coupled phenomena expected of any other quantum field theory. The magic of supersymmetry, however, is that in many cases we are able to make exact statements about the properties of the theory. This is because supersymmetry places certain restrictions on the kind of dynamics that can occur. Fortuitously, it turns out that these restrictions are not strong enough to stop interesting things happening, but are strong enough to allow us to solve certain aspects of the theory. In this way, supersymmetric field theories provide an important collection of toy models that allow us to understand what quantum field theory can do in regimes where we would otherwise have very little control.

Here is an example. The theory of the strong nuclear force, QCD, exhibits a remarkable property known as *confinement*. Quarks are always trapped inside hadrons and we never see isolated quarks on their own. There is no doubt that the theory of QCD has this property – we can see it clearly in numerical simulations – but we are a long way from being able to prove confinement from first principles. However, there are supersymmetric gauge theories, similar to QCD but with slightly different matter content, where confinement can be proven analytically. (This follows from the famous Seiberg-Witten solution of $\mathcal{N} = 2$ supersymmetric theories.) While the supersymmetric proof of confinement is not directly applicable to real-world QCD, it nonetheless gives us good intuition for how confinement might proceed in that context.

These lectures will very much be given in the spirt of using supersymmetry to tell us interesting things about strongly coupled quantum field theories. We will learn about topics that exist for real word QCD, such as confinement and chiral symmetry breaking, and see how these manifest themselves in more tractable supersymmetric theories. We will also learn about novelties that appear not to be of relevance for QCD but give us an insight into what strongly interacting quantum field theories can do. Foremost among these novelties is the concept of *duality*, the idea that two very different looking quantum field theories may, in fact, describe the same physics.

Reason 2: Mathematics

As our understanding of supersymmetric field theories grew, increasingly sophisticated mathematical constructs were found lurking within them. These are primarily, but not exclusively, ideas from geometry.

This link between supersymmetry and mathematics starts with some simple quantum mechanical models whose solutions give new perspectives on, among other things, Morse theory and index theorems. But the real fun starts when we turn to supersymmetric field theories. Understanding supersymmetric field theories in $d = 1 + 1$ dimensions led to the discovery of *mirror symmetry*, a relationship between topologically distinct manifolds. As we move to higher dimensional quantum field theories, we find ever more elaborate structures, some of which are known to mathematicians and some of which are novel. It is clear that there is much more to uncover.

We won't have anything to say about the connection to mathematics in these lectures, although we will stumble upon the concept of *Kähler geometry* as we proceed which at least gives a feel for how interesting geometric concepts arise naturally from supersymmetry. The companion lectures on Supersymmetric Quantum Mechanics have more of an eye towards the mathematical aspects of supersymmetry, albeit without getting very deep into the subject.

Reason 3: Our World

The million dollar question is: does supersymmetry have anything to do with our world? The rather disappointing answer is: we don't know.

There is certainly no experimental evidence that supersymmetry is a symmetry of nature at the fundamental level. Moreover, it's not for want of trying. To fill in the details, I'll first explain what it would mean for our world to be supersymmetric. Then I'll explain what reasons we have (or had!) for thinking that this might be the case.

In any supersymmetric theory, particles come in pairs – one a boson, the other a fermion – and this pair of particles share many of their properties, such as their masses and the forces that they experience. You don't need to build an LHC to realise that our world most certainly does not have this property! There is no bosonic particle with the same mass and charge as an electron; no massless fermionic particle with the same properties as the photon. (No, the neutrino doesn't do it!) There is, in short, no supersymmetry.

However, not all symmetries are manifest in the world around us. This is because of the phenomenon of *symmetry breaking* in which the dynamics of the theory make a choice which masks the underlying symmetry. There are many examples of symmetry breaking that we know take place, some mundane and familiar, others more exotic. Here are two. In a magnet, all the spins align in a given direction, breaking the underlying rotation symmetry. In the Standard Model, electroweak symmetry is broken by the Higgs boson ensuring, among other things, that the (left-handed) electron and neutrino look very different to our low-energy eyes despite the fact that they are indistinguishable at high energies.

It may well be that supersymmetry is a symmetry of our world but is broken and so hidden at low-energies. If this is the case, the breaking comes with an energy scale that we will call $M_{\rm susy}$. All of the superpartners – the other half of each boson/fermion pair – would get a mass that sits somewhere around $M_{\rm susy}$. So to answer the question of whether supersymmetry exists in nature we must also address the partner question: what is the scale of $M_{\rm susy}$?

For many years, supersymmetry was viewed as the most promising candidate for physics beyond the Standard Model, with $M_{\rm susy} \approx 1$ TeV. At this scale, supersymmetry provides a compelling solution to the hierarchy problem (the question of why the Higgs mass is not driven to higher scales by quantum fluctuations). Furthermore, if you adopt this solution then it comes with a number of happy consequences, from the unification of coupling constants to enticing candidates for dark matter.

However, with the advent of the LHC we have now explored the TeV scale and there is no sign of the predicted superpartners. It's not quite game over: it may well be that these extra particles are lurking just around the corner, tantalisingly out of reach of our current accelerator and will be found as we go to higher energies. But it's certainly fair to say that the parameter space of allowed theories has shrunk dramatically, as have our reasons for believing in supersymmetry at the TeV scale. This means that if supersymmetry is a symmetry of our world, it now appears to be broken at some scale $M_{\rm susy} \gtrsim 1$ TeV. But where?

There is reason to think that supersymmetry might show up by the time we reach the Planck scale $M_{\rm pl} \approx 10^{15}$ TeV. This reason is string theory. Of course, we don't know that string theory is the right theory of quantum gravity but it is presently the only viable candidate where a microscopic quantum theory gives the Einstein equations emerging at large distances. And string theory appears to require supersymmetry. (I include the word "appears" here because there are some open questions about bosonic (i.e. non-supersymmetric) string theory that we don't have a good handle on and it may be premature to throw this out as a viable theory.)

So if you buy into string theory, then you'll most likely want supersymmetry to be manifest by the time you get $M_{\rm pl}$. And, as we've seen above, it looks like it should be broken at some scale $M_{\rm susy} \gtrsim 1$ TeV. But there are 15 orders of magnitude between the TeV scale and the Planck scale. Where in this range should we expect supersymmetry to be broken if not at the TeV scale, or just above it, to provide a solution to the hierarchy problem? Sadly, I don't think that we have any good idea, and there are no hints from nature that it is more useful to have $M_{\rm susy}$ at some large scale $\gg$ TeV rather than another.

This leaves us with the current situation, one of no small befuddlement about what role, if any, supersymmetry has to play in our world. Given this, in these lectures we won't make any attempt to describe how supersymmetry may appear in our world. In particular, we will not devote effort to constructing supersymmetric versions of the Standard Model (the simplest is known as the MSSM where the first M stands for "minimal" and you can guess the rest) nor will we describe the many subtleties that come with how supersymmetry might be broken and how this manifests itself. Instead we will focus on places where supersymmetry has proved invaluable, viewing the theories as toy models to guide us in our understanding of quantum field theories.

1.1 A First Look at Supersymmetry

To motivate some of what lies ahead, we'll jump in with a particularly simple supersymmetric theory. The theory consists of a single, complex scalar ϕ together with a 2-component Weyl fermion ψ_α. (If you're unfamiliar with Weyl fermions, we'll describe their properties in detail in Section 2.1.)

The following action has kinetic terms for these two fields, together with some carefully tuned interactions

$$S = \int d^4x\ \left[\partial_\mu\phi^\dagger\partial^\mu\phi - i\psi\sigma^\mu\partial_\mu\bar{\psi} - \left|\frac{\partial W}{\partial\phi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial\phi^{\dagger\,2}}\bar{\psi}\bar{\psi}\right] \qquad (1.1)$$

Here $\sigma^\mu = (1, \sigma^i)$ with σ^i the usual collection of three Pauli matrices. Note that there is a relation between the scalar potential $V(\phi) = |W'(\phi)|^2$ and the scalar-fermion interactions, both of which are dictated by a function $W(\phi)$ known as the *superpotential*. If we want a renormalisable theory, this function should be no more than cubic

$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}\lambda\phi^3$$

This ensures that the potential is a quartic polynomical, $V(\phi) = |m\phi + \lambda\phi^2|^2$, while the scalar-fermion interactions take the usual Yukawa form $\phi\psi\psi$. Crucially, the function $W(\phi)$ should be *holomorphic*: it depends only on ϕ and not on $\phi^\dagger$. This fact will take on increasing significance as these lectures progress, but for now we will just take this as given.

Even without doing any detailed calculations, we can see that there's something curious about the action (1.1): the boson ϕ and the fermion ψ have the same mass $|m|$. Usually in quantum field theory, we shouldn't ascribe too much meaning to such an observation since masses receive quantum corrections and there's no guarantee that the physical masses of two distinct particles will coincide just because the masses in the Lagrangian are equal. However, for the particular action (1.1), it turns out that the equality of bosonic and fermionic masses persists in the full quantum theory. This arises because the action enjoys a rather surprising symmetry, with the infinitesimal variation given by

$$\delta\phi = \sqrt{2}\epsilon\psi \quad \text{and} \quad \delta\psi = \sqrt{2}i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi - \sqrt{2}\epsilon\frac{\partial W^\dagger}{\partial\phi^\dagger} \tag{1.2}$$

This is our first example of *supersymmetry*. It is a symmetry that relates the bosonic field ϕ with the fermionic field ψ. Because ψ is a Grassmann field, while ϕ is not, the infinitesimal object ϵ, which parameterises the transformation, must also be a Grassmann-valued Weyl spinor.

You can't tell just by staring at the action (1.1) that it is invariant under the supersymmetry transformation (1.2). Instead, it takes a calculation, one that turns out to be a little bit of a headache. (Some balm for this headache will be offered in Section 3.2.3.)

The action (1.1) is the simplest supersymmetric theory in $d = 3 + 1$ dimensions. It is known as the *Wess-Zumino* model. The existence of such a symmetry opens up a number of questions. What, if anything, is the symmetry good for? Are there other theories that also exhibit such symmetry? What properties might they have? All of these will be answered as these lectures progress.

There is also another question that might have occurred to you: why is it such a pain to see that the action (1.1) is invariant under supersymmetry? Usually, the existence of symmetries in an action jumps out at you. Indeed, one of the main advantages of working with the Lagrangian approach, rather than the Hamiltonian approach, is that all symmetries are manifest. Typically you need do little more than ensure that various indices are contracted in the right way. This suggests that there may be a better way to write the action (1.1) that makes supersymmetry as obvious as any other symmetry. And there is. Our first task in these lectures – one that will carry us through much of Sections 2, 3 and 4 – is to better understand the structure behind supersymmetry and the corresponding supersymmetric actions.