Berry Phase and Supersymmetry

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The Project

- Study non-Abelian Berry phase in string theory and other supersymmetric systems.

- Various results:
  - Exact results on Berry phase in strongly coupled systems
  - D0-branes: anyons, Hopf maps
  - Gravitational Precession and AdS/CFT

  - Based on work with Julian Sonner
    - arXiv:0809.3783 and 0810.1280
  - Also earlier work with Julian and Chris Pedder

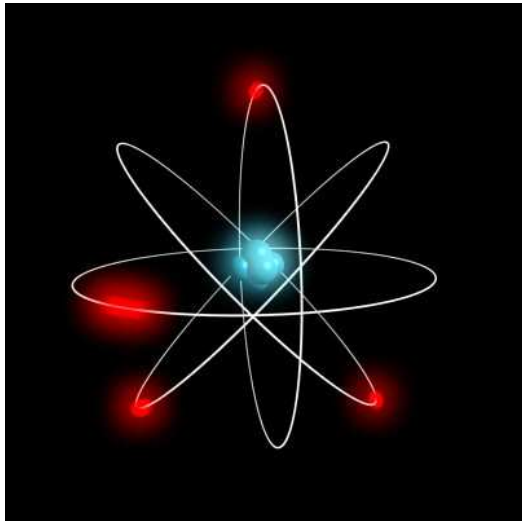
Review of Non-Abelian Berry Phase

- Berry, Wilczek and Zee

Berry Phase

Parameters of the Hamiltonian

Hamiltonian



- Prepare system in an energy eigenstate.
- Change parameters slowly. Adiabatic theorem ensures that the system clings to the energy eigenstate (as long as we avoid degeneracies).

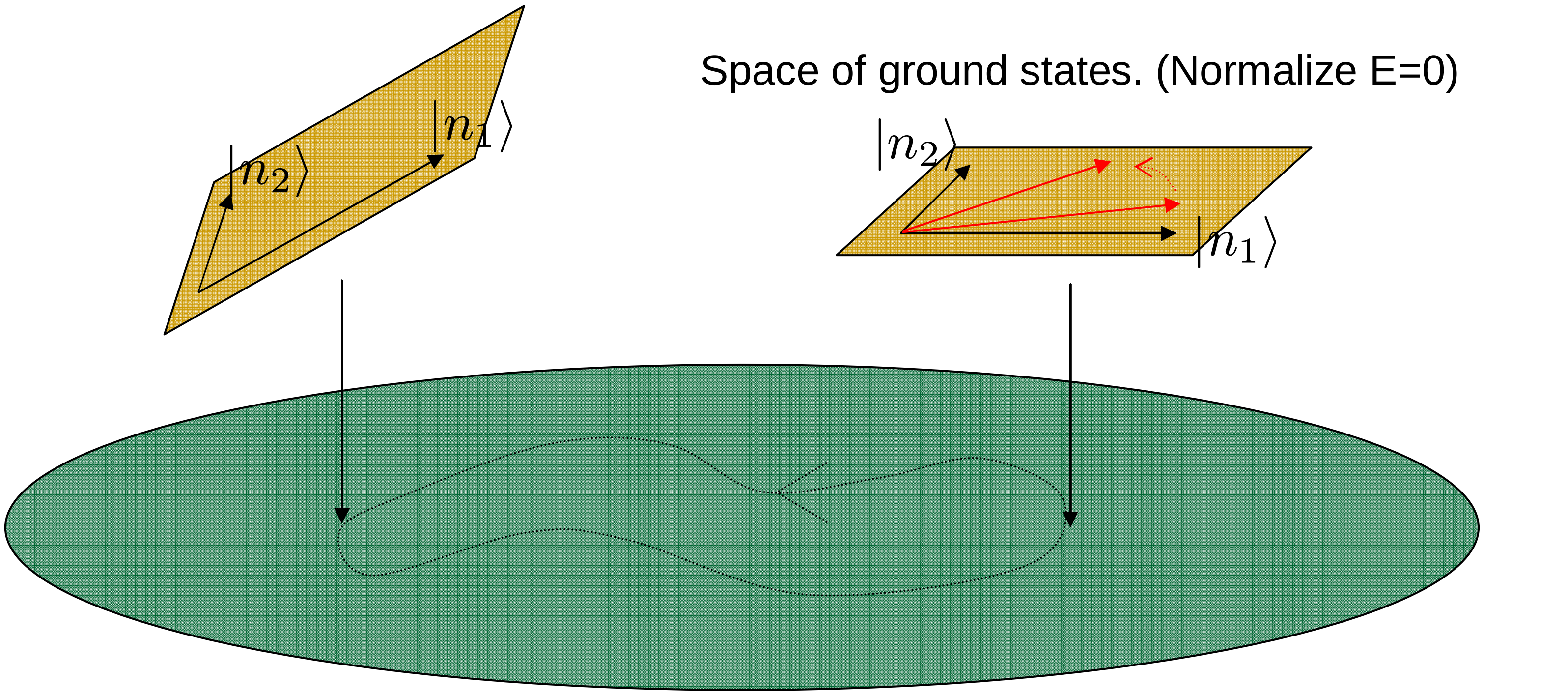
Berry Phase

- Question: Perform a loop in parameter space. The original state will come back to itself *up to a phase.* What is this phase?

- Answer: There is the usual dynamical phase e^{-iEt}. But there is another contribution that is independent of time, but depends on the path taken in parameter space. This is Berry's phase.

- There is also a non-Abelian generalization. Suppose that the energy eigenstate is n-fold degenerate for all values of the parameters. The state now comes back to itself up to a U(n) rotation. (Wilzcek and Zee).

Non-Abelian Berry Phase



For each $\vec{\lambda}$, introduce an arbitrary set of bases for the ground states:

$$|n_a(\vec{\lambda})\rangle \qquad a = 1, \ldots, n$$

Non-Abelian Berry Phase

We want to know the evolution of the state $|\psi(t)\rangle$ under

$$i\partial_t|\psi\rangle = H(\vec{\lambda}(t))\,|\psi\rangle = 0$$

We write $|\psi_a(t)\rangle = U_{ab}(t)\,|n_b(\vec{\lambda}(t))\rangle$ (which assumes the adiabatic theorem)

$$\Longrightarrow \qquad |\dot{\psi}_a\rangle = \dot{U}_{ab}|n_b\rangle + U_{ab}|\dot{n}_b\rangle = 0$$

$$\Longrightarrow \qquad \begin{aligned} U^\dagger_{ac}\dot{U}_{ab} &= -\langle n_a|\dot{n}_b\rangle \\ &= -\langle n_a|\partial_{\vec{\lambda}}|n_b\rangle \cdot \dot{\vec{\lambda}} \\ &\equiv -i\vec{A}_{ba}\cdot\dot{\vec{\lambda}} \end{aligned}$$

Non-Abelian Berry Connection

The rotation of the state after a closed path is given by

$$\boxed{U = P \exp\left(-i \oint \vec{A} \cdot d\vec{\lambda}\right)}$$

where $\vec{A}_{ba}$, a Hermitian u(n) connection over the space of parameters, is given by

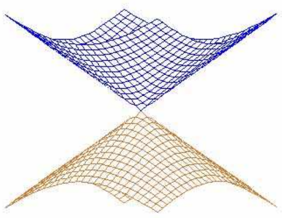
$$\boxed{\vec{A}_{ba} = -i \langle n_a | \partial_{\vec{\lambda}} | n_b \rangle}$$

Note: this is really a connection. Changing the basis at each point, changes the connection by

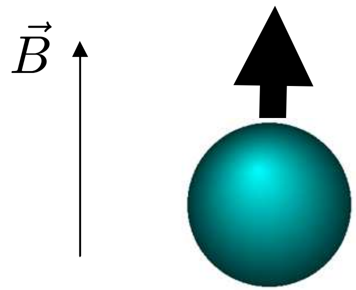
$$|n'_a(\vec{\lambda})\rangle = \Omega_{ab}(\vec{\lambda})\, |n_b(\lambda)\rangle \quad \Longrightarrow \quad \vec{A}' = \Omega \vec{A} \Omega + i(\partial_{\vec{\lambda}} \Omega)\, \Omega^{\dagger}$$

An Example of Abelian Berry Phase

The canonical example of Berry phase is a spin ½ particle in a magnetic field



$$H = -\vec{B}\cdot\vec{\sigma}$$



It's easy to write down the ground states and
compute the Berry connection

$$A_\psi = \frac{1-\cos\theta}{2B\sin\theta}$$

$$B_1 = B\sin\theta\sin\psi$$
$$B_2 = B\sin\theta\cos\psi$$
$$B_3 = B\cos\theta$$

$$\Longrightarrow \quad \epsilon_{ijk}F_{jk} = \frac{B_i}{2B^3}$$

This is the Dirac monopole!

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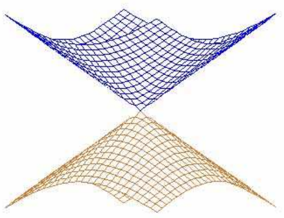

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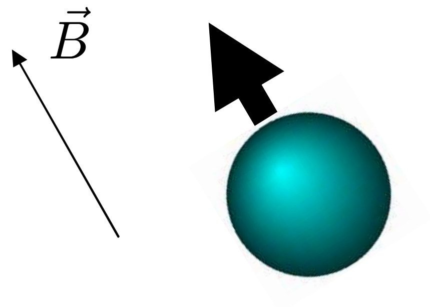

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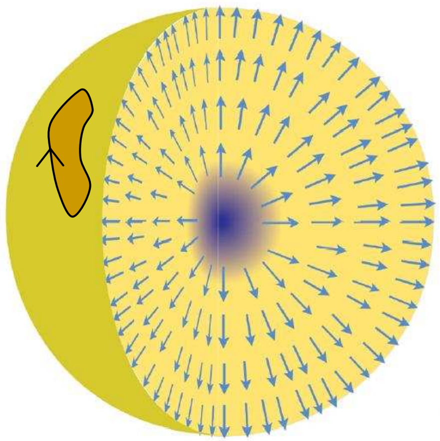
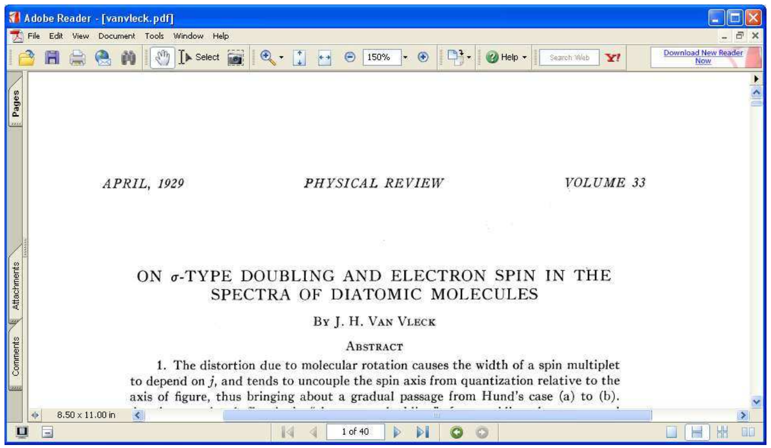
The Magnetic Monopole

- This is a gauge connection over the space of magnetic fields…confusing!
- The singularity at B=0 reflects the fact that the two states are degenerate at this point. (Our "Wilsonian effective action" breaks down).
- Moving on a path that avoids the singularity gives rise to a phase $\exp(-i \int dS \cdot F)$. The physics is dictated by the singularity, even though we steer clear of it!
- The magnetic monopole first appeared in the context of Berry's phase. (Two years before Dirac's paper!)





APRIL, 1929 PHYSICAL REVIEW VOLUME 33

ON σ-TYPE DOUBLING AND ELECTRON SPIN IN THE SPECTRA OF DIATOMIC MOLECULES

By J. H. Van Vleck

Abstract

1. The distortion due to molecular rotation causes the width of a spin multiplet to depend on j, and tends to uncouple the spin axis from quantization relative to the axis of figure, thus bringing about a gradual passage from Hund's case (a) to (b).

Berry Phase in Susy Quantum Mechanics

- Exact Results for Berry Phase
- Based mostly 0809.3783 and 0810.1280

Part 1: Susy Quantum Mechanics

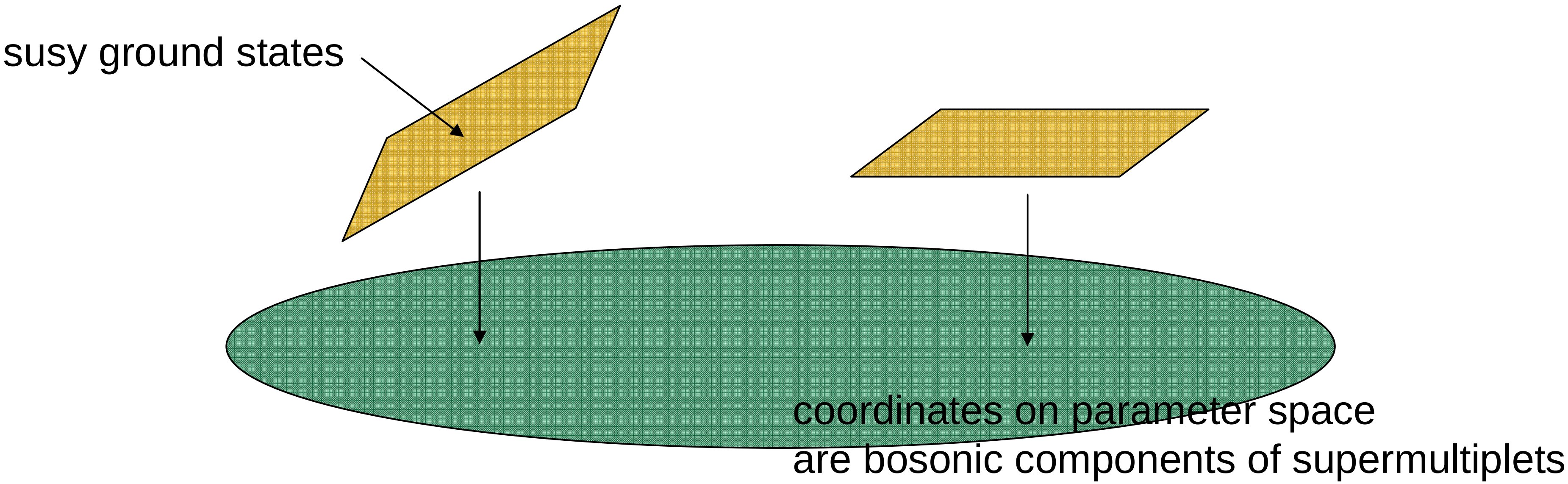
- N=(2,2) Susy QM is the dimensional reduction of N=1 theories in four dimensions. (Sometimes called N=4 or N=4a)

Why Berry Phase?

- Witten index implies multiple ground states
  - Naturally get non-Abelian connections
- Berry phase is the BPS quantity!
- We will show that the Berry connection must satisfy certain equations.

Supersymmetric Parameters

- **Key point:** Parameters can be thought of as the lowest (bosonic) components of supermultiplets
  - complex parameters of the superpotential live in chiral multiplets
  - triplets of parameters live in vector multiplet



Supersymmetric Holonomy

$$U = T \exp\left(-i \oint \vec{A} \cdot \dot{\vec{\lambda}}\, dt\right)$$

- The holonomy should itself be supersymmetric. Generalise to…

$$U = T \exp\left(-i \oint \mathcal{A}\, dt\right)$$

with $\mathcal{A} = \vec{A} \cdot \dot{\vec{\lambda}} + \text{ susy completion}$

- $\mathcal{A}$ is a U(N) valued object. To be invariant under susy, we require

$$\delta \mathcal{A} = \frac{d\Theta}{dt} + i[\mathcal{A}, \Theta]$$

Chiral Multiplets: (ϕ, ψ_α, F)

$$\mathcal{A} = A(\phi, \phi^\dagger)\dot{\phi} + G(\phi, \phi^\dagger)F + B(\phi, \phi^\dagger)\psi\psi + C(\phi, \phi^\dagger)\bar{\psi}\psi$$

- A, G, B and C are all NxN matrices. Susy requires

$$C = [D, D^\dagger] = -[G, G^\dagger]$$

$$B = DG \quad \text{and} \quad D^\dagger G = 0 \quad \text{where} \quad D = \frac{\partial}{\partial \phi} + i[A, \cdot]$$

- These are the Hitchin equations. G has the interpretation of a particular correlation function in the QM

- Multiple chiral multiplets give the tt* equations
  - c.f. Cecotti and Vafa

Vector Multiplets: $(A_0, \vec{X}, \chi_\alpha, D)$

$$\mathcal{A} = \vec{A}(X) \cdot \dot{\vec{X}} - H(X)D + \vec{C}(X) \cdot \bar{\chi}\,\vec{\sigma}\,\chi$$

- A, H and C are all NxN matrices. Susy requires

$$C_i = D_i H \equiv \frac{\partial H}{\partial X^i} + i[A_i, H]$$

$$F_{ij} = \epsilon_{ijk} \mathcal{D}_k H$$

- These are the Bogomolnyi equations. Their solutions describe BPS monopoles. Again, H is a correlation function in the QM

- Multiple vector multiplets give a generalization of these equations.

An Example: Spin ½ Particle on a Sphere

$$H = -\frac{\hbar^2}{2m}\nabla^2\,1_2 - \hbar\vec{B}\cdot\vec{\sigma}\,\cos\theta + \frac{1}{2}B^2\sin^2\theta\,1_2$$

Magnetic field

Potential

- This is a truncation of the N=(2,2) CP^1 sigma-model.
- The triplet of parameters B sit in a vector multiplet

- The theory has two ground states for all vaues of B
  - This means that we get a U(2) Berry connection $\mathbf{R}^3$

The Berry Connection

- The Berry connection must satisfy the Bogomolnyi equation

$$F_{ij} = \epsilon_{ijk}\mathcal{D}_k H$$

- For this particular model, H is the correlation function

$$H^a_b = \langle a|\cos\theta|b\rangle$$

- This means that we can just write down the answer

$$A_i = \epsilon_{ijk}\frac{B_j\sigma^k}{2B^2}\left(1 - \frac{2Bm/\hbar}{\sinh(2Bm/\hbar})\right)$$

- Important point: the Berry connection is smooth at the origin

Technical Aside

- The 't Hooft-Polyakov connection has an expansion

$$A_i = \epsilon_{ijk} \frac{B_j \sigma^k}{2B^2} \left(1 - \frac{4Bm}{\hbar} e^{-2Bm/\hbar} + \ldots\right)$$

1-instanton effect

- 1-loop determinants around the background of the instanton are non-trivial. (c.f. 3 dimensional field theories)

- Higher effects are instanton-anti-instanton pairs
  - i-ibar pairs *can* contribute to BPS correlation functions

Summary of Part 1

- Summary:
  - Non-Abelian Berry connection is BPS quantity.
  - Exact results are possible, exhibiting interesting and novel behaviour

- Applications and Future work
  - Mathematical: equivariant cohomology and curvature of bundles of harmonic forms
  - Quantum Computing: connections for non-Abelian anyons in FQHE states
  - Black Holes:
    - Relationship to attractor flows (c.f. de Boer et al.)
    - Entanglement of black holes in Denef's quantum mechanics

Part 2: Berry Phase of D0-Branes

- Based on arXiv:0801.1813

The Berry Phase of D0-Branes

- Consider SU(2) Susy quantum mechanics with N=2,4,8 and 16 supercharges.
- This describes relative motion of two D0-branes in d=2,3,5 and 9 spatial dimensions.

$$L = \frac{1}{2g^2}\,\mathrm{Tr}\left((\mathcal{D}_0 X_i)^2 + \sum_{i<j}[X_i, X_j]^2 + i\bar{\psi}\mathcal{D}_0\psi + \bar{\psi}\Gamma^i[X_i, \psi]\right)$$

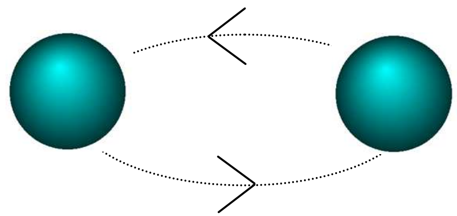
$$\{\Gamma^i, \Gamma^j\} = 2\delta^{ij} \quad i, j = 1, \ldots, d$$

- Work in the Born-Oppenheimer approximation
- Separate D0-branes so that SU(2) breaks to U(1)
- Construct the Hilbert space for excited strings.

- Question: How does Hilbert space evolve as D0-branes orbit?

D0-Branes in the Plane

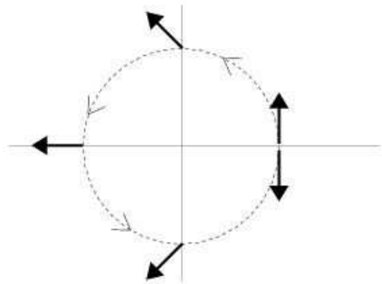
- With N=2 supercharges, the D0-branes move on the plane
  - They are fractional D0-branes, trapped at a singularity of a G2-holonomy manifold
- The Berry phase of the ground state is a minus sign picked up after a single orbit



$$\begin{pmatrix} X^2 & X^1 \\ X^1 & -X^3 \end{pmatrix} \vec{\lambda} = -\vec{\lambda}$$

- This means that after the *exchange* of particles, the wavefunction changes by

$$|\Omega\rangle \to \pm i |\Omega\rangle$$



- The D0-branes are *anyons*!
- U(N) matrix model is a description of a gas of N anyons.

D0-Branes in Three Dimensions

- With N=4 supercharges, the D0-branes move in d=3 spatial dimensions
  - They are trapped at a CY singularity
- The ground state does not feel a Berry phase
- Excited states do: the Berry phase is the Dirac monopole. The effective motion of the D0-branes is governed by

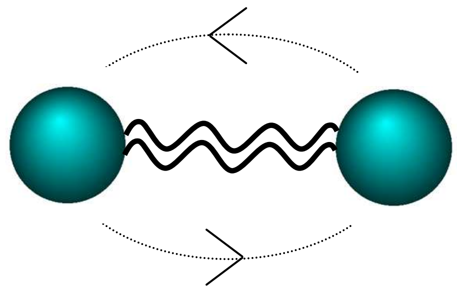
$$L \sim \frac{1}{g^2}\dot{X}_i^2 + A_i^{\mathrm{Dirac}}(X)\dot{X}_i - |X|$$

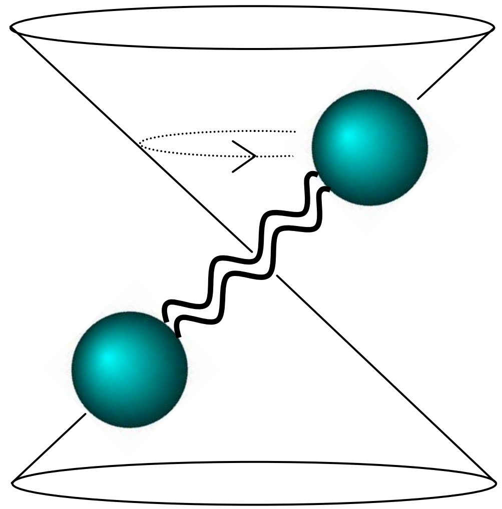
- The states follow non-relativistic Regge trajectories

$$E^3 \sim g^2 J(J+q)$$

Angular Momentum, J

Dirac monopole charge, q=1





The N=8 and N=16 Theories

- The D0-branes move in d=5 and d=9 respectively
- The excited states are degenerate, multiplets of R-symmetry
- The Berrry phase is non-Abelian

$$|\psi_a\rangle \longrightarrow P\exp\left(-i\oint \vec{A}\cdot d\vec{X}\right)_{ab}|\psi_b\rangle$$

- SU(2) Yang monopole for N=8

$$\frac{1}{8\pi^2}\int_{S^4}\mathrm{Tr}F\wedge F = +1$$

- SO(8) Octonionic Monopole for N=16

$$\frac{1}{4!(2\pi)^4}\int_{S^8}\mathrm{Tr}F\wedge F\wedge F\wedge F = +1$$

Berry, Hopf and Supersymmetry

There is a nice relationship appearing here between supersymmetry, the four division algebras, and the Hopf maps

$$S^1 \xrightarrow{Z_2} S^1 \cong \mathbf{RP}^1$$

$$S^3 \xrightarrow{S^1} S^2 \cong \mathbf{CP}^1$$

$$S^7 \xrightarrow{S^3} S^4 \cong \mathbf{HP}^1$$

$$S^{15} \xrightarrow{S^7} S^8 \cong \mathbf{OP}^1$$

+/-1	U(1) Dirac Monopole	SU(2) Yang Monopole	SO(8) Octonionic Monopole

These non-Abelian Berry phases have appeared in the condensed matter literature: Zhang et al, Bernevig et al.

Summary of Part 2

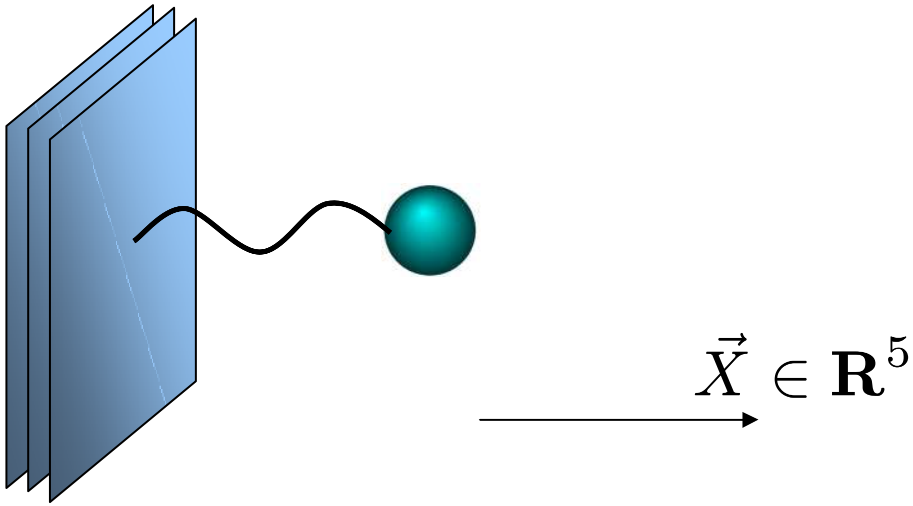
- N=2 Susy: Sign Flip
  - anyons
- N=4 Susy: Dirac Monopole
  - deformed Regge trajectories
- N=8 Susy: SU(2) Yang Monopole
- N=16 Susy: SO(8) Octonionic Monopole
  - hint at octonionic structure?

Part 3: Berry Phase and AdS/CFT

- Based on arXiv:0709.2136

N=8 Susy Quantum Mechanics

The D0-D4-Brane System
in IIA String Theory



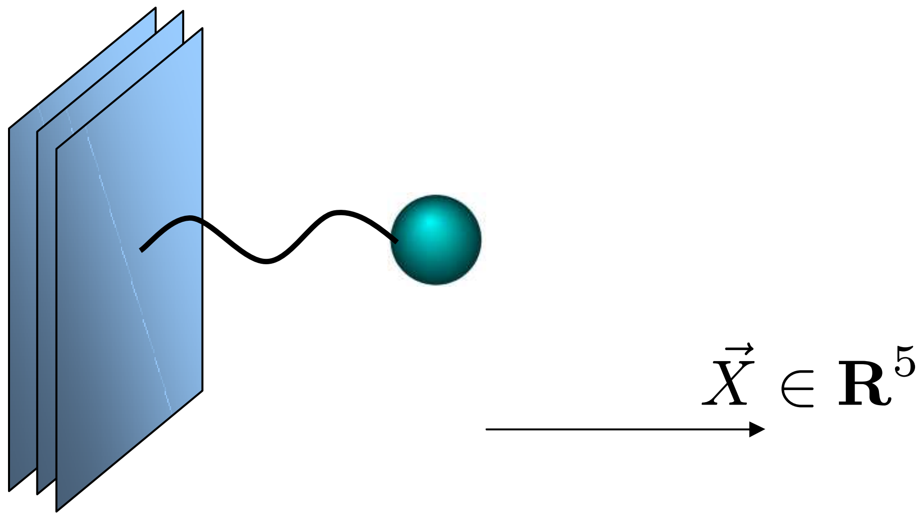
$$\mathcal{L}_{D0} = \frac{1}{g^2}(\dot{\vec{X}}^2 + \bar{\lambda}\dot{\lambda}) + \sum_{i=1}^{N} |\mathcal{D}_t \phi_i|^2 + |\mathcal{D}_t \tilde{\phi}_i|^2 + \bar{\psi}_{\alpha i}\mathcal{D}_t \psi_{\alpha i}$$

$$- \sum_{i=1}^{N} \vec{X}^2(|\phi_i|^2 + |\tilde{\phi}_i|^2)| + \bar{\psi}_{\alpha i}(\vec{X} \cdot \vec{\Gamma}_{\alpha\beta})\psi_{\alpha i}$$

$$+\text{Yukawa} + \text{Potential}$$

with Γ_a five 4×4 matrices such that $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$

Born-Oppenheimer Approximation



- Make D0-branes heavy: $g^2 \to 0$
- Treat X as a fixed parameter, and quantize the D0-D4 strings
- Integrate out the D0-D4 strings to write an effective action for X

Quantizing the Fermions

$$\{\psi_\alpha, \bar{\psi}_\beta\} = \delta_{\alpha\beta} \implies$$ Creation and Annihilation Operators

$$\alpha = 1, \ldots, 4$$

Define $|0\rangle$ such that $\psi_\alpha|0\rangle$. Then the fermionic sector of D0-D4 strings gives

	Multiplicity		Eigenvalues of $H_F = \bar{\psi}\vec{X} \cdot \vec{\Gamma}\psi$
$\lvert 0\rangle$	1		0
$\bar{\psi}_\alpha \lvert 0\rangle$	4		-X,-X,+X,+X
$\bar{\psi}_\alpha\bar{\psi}_\beta \lvert 0\rangle$	6	Ground State	-2X,0,0,0,0,+2X
$\bar{\psi}_\alpha\bar{\psi}_\beta\bar{\psi}_\gamma \lvert 0\rangle$	4		-X,-X,+X,+X
$\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 \lvert 0\rangle$	1		0

Quantizing the Fermions

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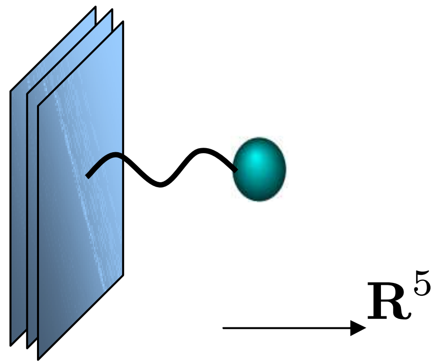
$\bar{\psi}_\alpha\bar{\psi}_\beta\bar{\psi}_\gamma\lvert 0\rangle$	4	-X,-X,+X,+X
$\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4\lvert 0\rangle$	1	0

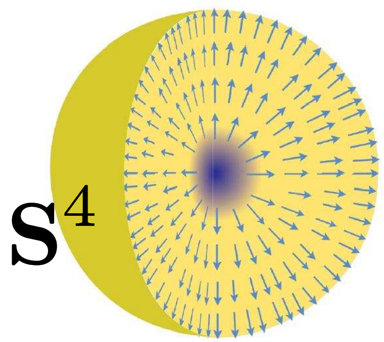
Interesting Excited States

Berry Phase



Question: Sit in one of the two excited states in the sector $\bar{\psi}_\alpha|0\rangle$. Adiabatically move the D0-brane around the D4. What is the Berry phase?

Answer: It is the *SU(2) Yang Monopole*. This is a rotationally symmetric connection over R^5



$$\int_{\mathbf{S}^4} \mathrm{Tr} F \wedge {}^\star F = 8\pi^2$$

This has appeared before as Berry's phase in the condensed matter literature.

Yin and Yang

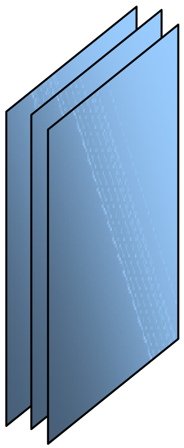
- In the $\bar{\psi}_\alpha|0\rangle$ sector, the Berry connection is the Yang monopole.
- In the $\bar{\psi}_\alpha\bar{\psi}_\beta\bar{\psi}_\gamma|0\rangle$ sector, the Berry connection is the anti-Yang, or Yin, monopole.

$$A_\mu^{\text{Berry}} = \begin{pmatrix} A_\mu^{\text{Yang}} & 0 \\ 0 & A_\mu^{\text{Yin}} \end{pmatrix} \xrightarrow{\text{g.t.}} \frac{X_\nu \Gamma_{\nu\mu}}{X^2}$$

$$\Gamma_{\mu\nu} = \tfrac{1}{4i}[\Gamma_\mu, \Gamma_\nu]$$

Supergravity and Strong Coupling

- The previous analysis is valid when $g^2 N \ll X^3$
- When $g^2 N \gg X^3$ we can instead replace the D4-branes by their supergravity background*



$$ds^2 = H^{-1/2}(X)(-dt^2 + d\vec{y}^2) + H^{+1/2}(X)d\vec{X}^2$$

$$e^{2\phi} = H^{-1/2}$$

$$\boxed{H = 1 + \frac{g^2 N}{X^3}}$$

We now place a probe D0-brane in this background and read off the low-energy dynamics.

* Caveat: This isn't quite the usual AdS/CFT: there is a non-renormalization theorem at play here.

The D0-Brane Probe

- The low-energy dynamics of the D0-brane in the D4-brane background is given by

$$\mathcal{L}_{D0} = \tfrac{1}{2} H(X)\, \dot{\vec{X}}^2 + H(X)\, \bar{\lambda}_\alpha D_t \lambda_\alpha + \ldots$$

- The covariant derivative for the spinor is

spin connection

$$D_t \lambda_\alpha = \dot{\lambda}_\alpha + \dot{\vec{X}} \cdot \vec{\omega}_\alpha{}^\beta\, \lambda_\beta$$

- which ensures parallel transport of free spinors

Gravitational Precession

- The excited states that we studied at weak coupling carry the same quantum numbers as a spinning particle at strong coupling. The quantum Berry connection maps into *classical gravitational precession* of the spin.

- In the near horizon limit of the D4-branes,

$$H = 1 + \frac{g^2 N}{X^3} \to \frac{g^2 N}{X^3}$$

- The spin connection of the metric in the near horizon limit is

$$\omega_\mu = \frac{3}{2} \frac{X_\nu \Gamma_{\nu\mu}}{X^2}$$

- which differs by 3/2 from weak coupling result. (Smooth, or level crossing?)

Summary of Part 3

- Summary
  - Berry’s phase is associated to Yang Monopole.
  - Berry’s Phase in strongly coupled system is gravitational precession.

- Questions
  - Relation to six-dimensional (2,0) theory and Wess-Zumino terms?