Heterotic Vortex Strings

David Tong

UNIVERSITY OF CAMBRIDGE

The Take-Home Message

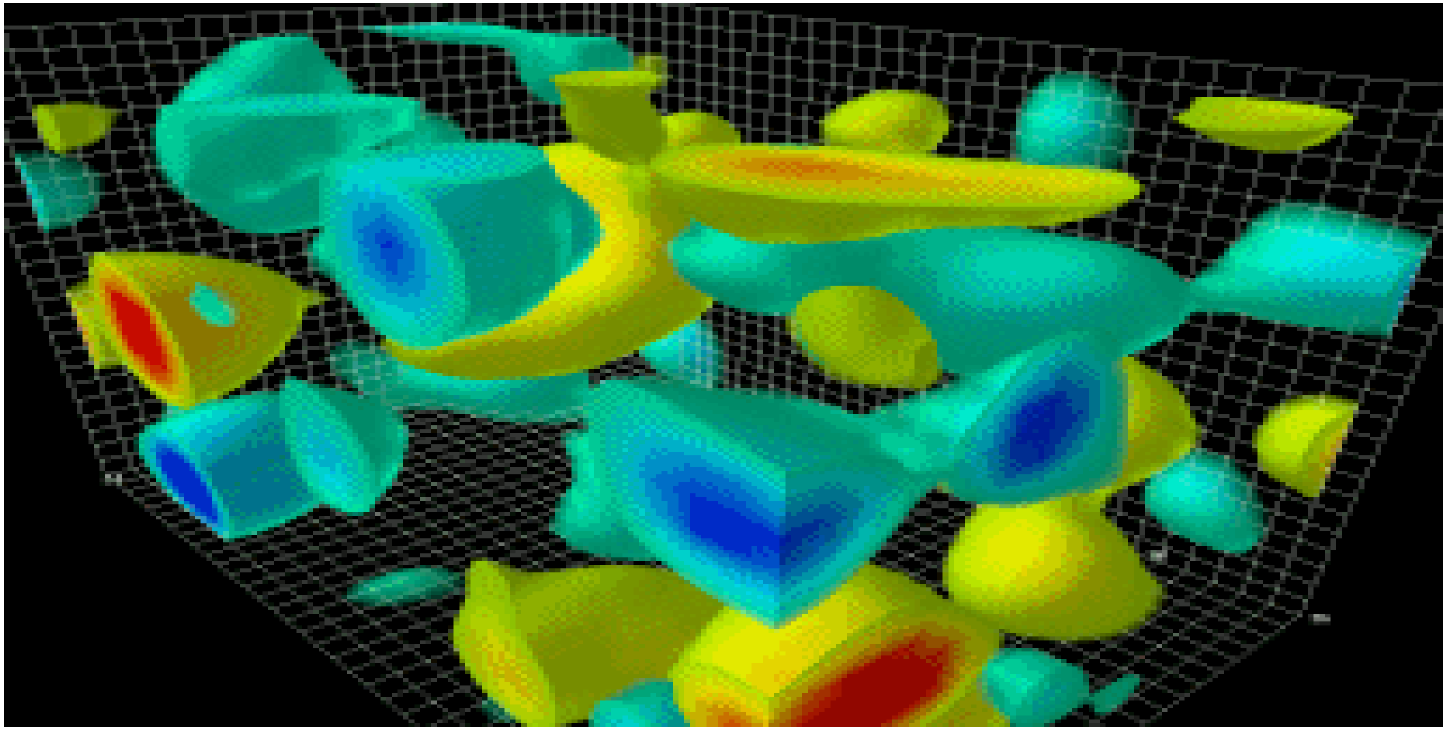
- For 25 years we've known that 4d non-abelian gauge theories share certain features with 2d sigma-models

  - Asymptotic freedom
  - Confinement
  - Dynamically generated mass gap
  - Anomalies
  - Instantons
  - Chiral Symmetry Breaking
  - Large N limits

- In fact, there are *quantitative* links between the two. The relationship is derived through the dynamics of solitonic vortex strings.

A Cartoon of the Basic Idea

- Take a strongly coupled theory with U(N) gauge group and some fundamental scalar fields.



Graphics by Leinweber

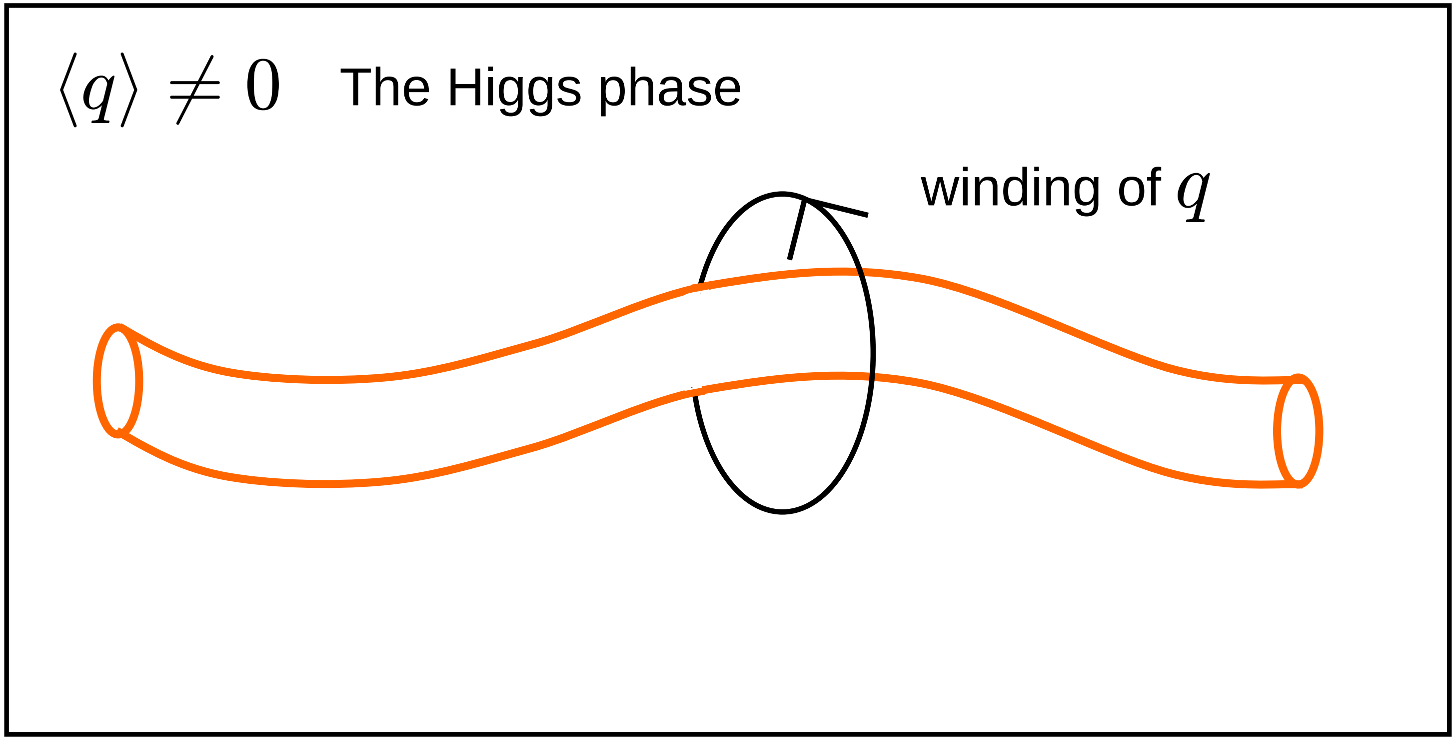
A Cartoon of the Basic Idea

- Deform the theory by inducing an expectation value for the scalar fields
  - If the gauge group is completely broken, the theory now lies in the weakly coupled Higgs phase

$\langle q \rangle \neq 0$ The Higgs phase

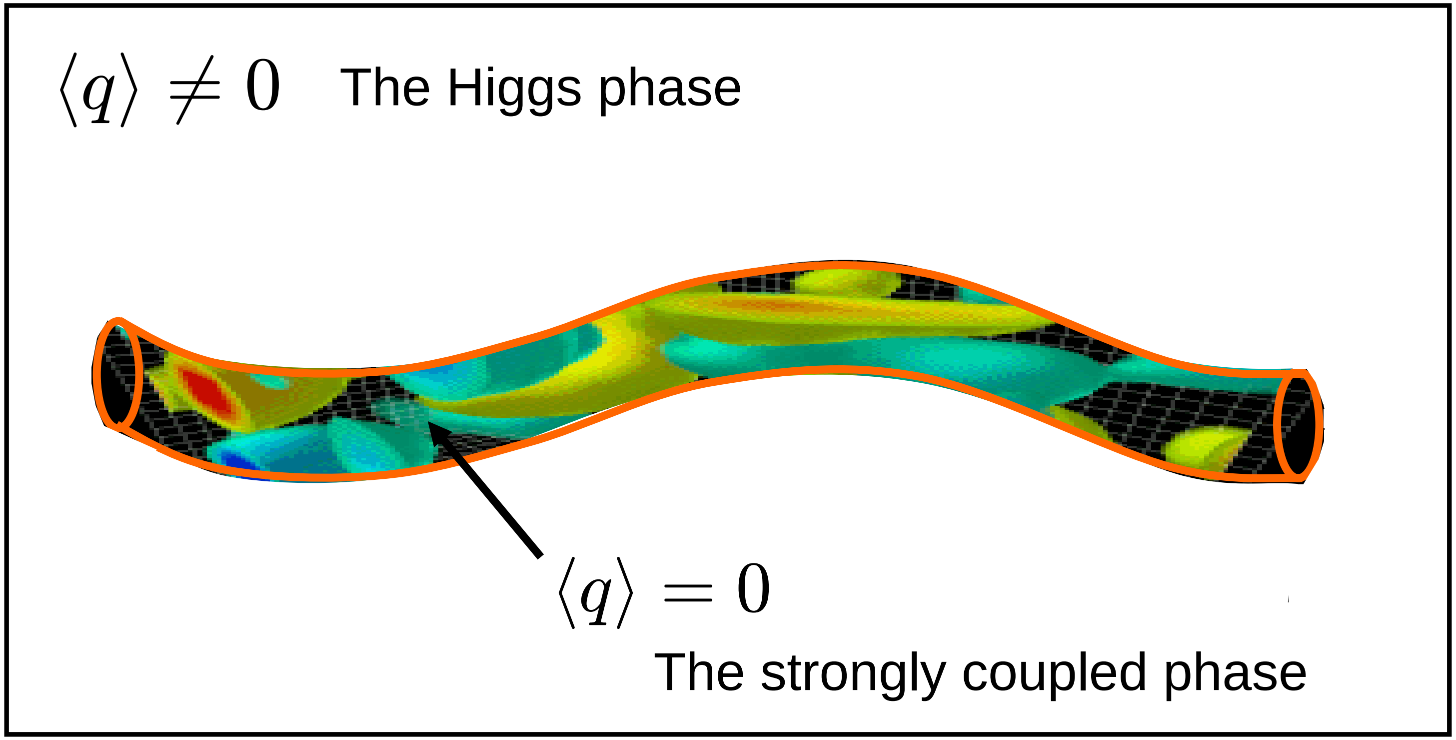
A Cartoon of the Basic Idea

- The theory now admits vortex strings, supported by the phase of the scalar winding at infinity



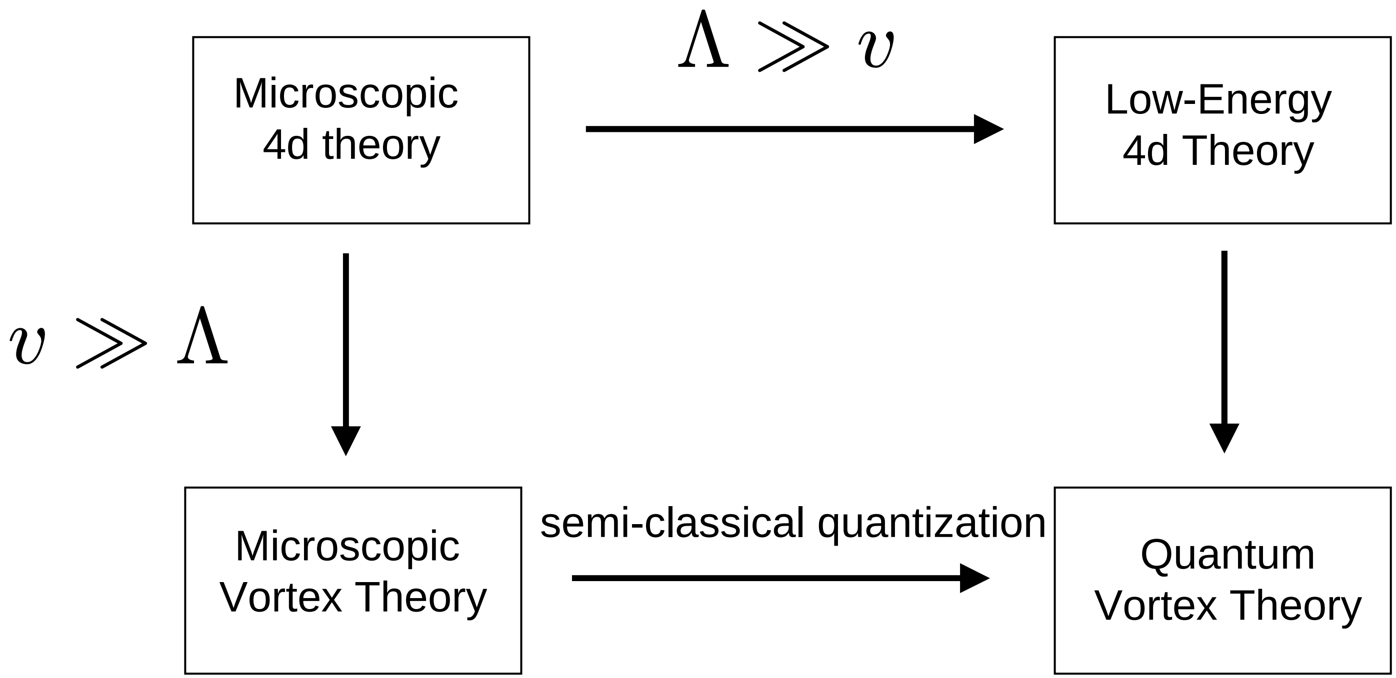
A Cartoon of the Basic Idea

- The interior of the vortex string is a strongly coupled system
  - The vortex string knows about the original 4d gauge theory.



Two Paths to Soliton Quantization

Two scales: Λ the strong coupling scale
v the symmetry breaking scale



The Results of this Technique

By studying the worldsheet theory of the vortex string, we can reconstruct the following results about four dimensional gauge theories:

$$\underline{\text{4d } \mathcal{N} = 2 \text{ Theories: } (N_f \geq N_c)}$$

- Exact BPS Mass Spectrum, including all quantum effects
- The Dimensions of Chiral Primary Operators at Superconformal Points
- The Seiberg-Witten Curve as the twisted superpotential of the worldsheet theory.

The Results of this Technique

$4\mathrm{d}\ \mathcal{N}=1$ Theories: $(N_f = N_c)$

- Quantum Numbers of Spectrum: i.e. Confinement
- $\det M - B\tilde{B} = \Lambda^{2N}$

The purpose of this talk is to describe these results for theories with N=1 supersymmetry: the vortex strings have N=(0,2) supersymmetry on their worldsheet. They are called *heterotic vortex strings.*

The Basic Theory

Starting point: d=3+1 with U(N) gauge group and N_f =N fundamental flavours.

$$L = \frac{1}{4e^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}q_i|^2$$
$$- \frac{e^2}{2}\,\mathrm{Tr}\,(\sum_i q_i \otimes q_i^\dagger - v^2)^2$$

We write ${q^a}_i$ where a=1,…,N is the colour index, and i=1,…,N_f is the flavour index.

The 4d Theory

- Vacuum: The ground state is unique (up to a gauge transformation)

$$q^a{}_i = v\,\delta^a{}_i$$

- Spectrum: The theory has a mass gap,with

$$m_\gamma = m_q \sim ev$$

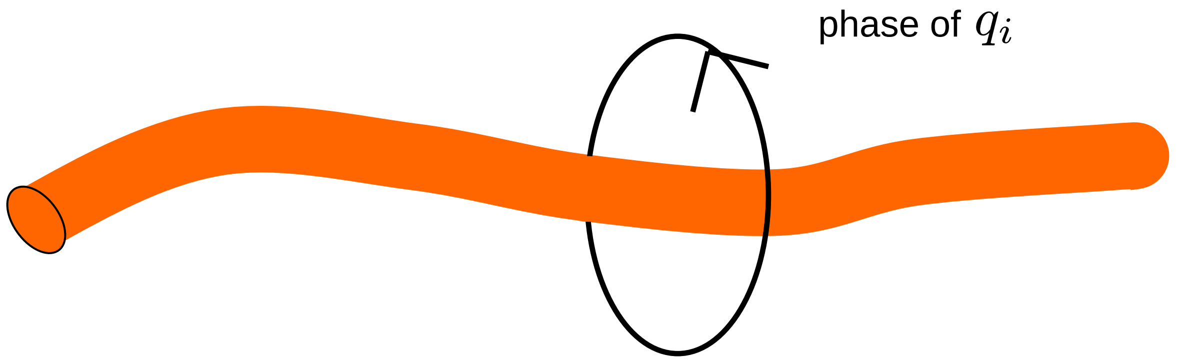
- Symmetries: The theory lies in the “colour-flavour” locked phase

$$U(N) \times SU(N_f) \rightarrow SU(N)_{\rm diag}$$

Note that overall U(1) is broken: $\Pi_1(U(1)) \cong \mathbf{Z}$ $\Longrightarrow$ Vortices

Vortices

Broken U(1) gauge symmetry $\Longrightarrow$ Vortices



$$(B_3)^a{}_b = e^2\left(\sum_i q_i^a q^\dagger_{ib} - v^2 \delta^a{}_b\right)$$

$$(\mathcal{D}_z q_i)^a = 0$$

$z = x_1 + i x_2$

$$\boxed{T_{\text{vortex}} = 2\pi v^2}$$

Nielsen and Olesen, '73

Vortex Moduli Space

Suppose we have an Abelian vortex solution $B_\star$,$q_\star$. We can trivially embed this in the non-Abelian theory.

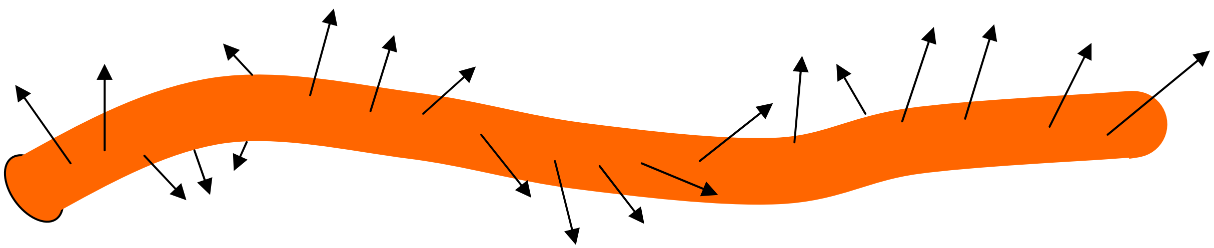
$$B = \begin{pmatrix} B_\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \qquad q = \begin{pmatrix} q_\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}$$

Different embeddings $\Longrightarrow$ moduli space of vortex

$$SU(N)_{\mathrm{diag}}/SU(N-1)\times U(1) \cong \mathbf{CP}^{N-1}$$

Vortex Dynamics

The low energy dynamics of an infinite, straight vortex string is the d=1+1 sigma model with target space $\mathbf{C} \times \mathbf{CP}^{N-1}$



Size of $\mathbf{CP}^{N-1}$ is $\boxed{r = \frac{2\pi}{e^2}}$

This means that when the 4d theory is weakly coupled, the 2d theory is also weakly coupled.

More General Vortex Dynamics

<u>4d Gauge Theory</u>	<u>2d Sigma Model</u>
U(N) Gauge Theory	CP^{N-1} Sigma-Model
Add Charged Fermions	Fermion Zero Modes
Add Charged Bosons	Further Bosonic Zero Modes
Change Scalar VEVs	Induce Potentials on Target Space
Add Interactions (e.g Yukawa)	Add Interactions (e.g. 4-fermi)

An Example: Supersymmetric Theories

N=2 4d theory $\Longrightarrow$ N=(2,2) sigma model
N=1 4d theory $\Longrightarrow$ N=(0,2) sigma model

$$L_{(0,2)} = \sum_{i=1}^{N} |\mathcal{D}\phi_i|^2 - D\left(\sum_i |\phi_i|^2 - r\right)$$

$$+2i(\bar{\psi}_{-i}\mathcal{D}_+\psi_{-i} + \bar{\psi}_{+i}\mathcal{D}_-\psi_{+i}) + \bar{\phi}_i\psi_{+i}\zeta_-$$

$$L_{(2,2)} = L_{(0,2)} + |\sigma|^2|\phi_i|^2 + \bar{\phi}_i\psi_{-i}\zeta_+ + \bar{\psi}_{-i}\sigma\psi_{+i} + \text{h.c}$$

$$\sum_i |\phi_i|^2 = r \bmod \phi_i \to e^{i\alpha}\phi_i \quad \Longrightarrow \quad \mathbf{CP}^{N-1}$$

Review of N=1 SQCD

$N_f = N_c$: The quantum deformed moduli space

$$V_{4d} = \frac{e^2}{2}\mathrm{Tr}(q_i q_i^\dagger - \tilde{q}_i^\dagger \tilde{q}_i)^2$$

Classically the theory has flat directions parameterized by

Mesons: $M_{ij} = \tilde{Q}_i Q_j$

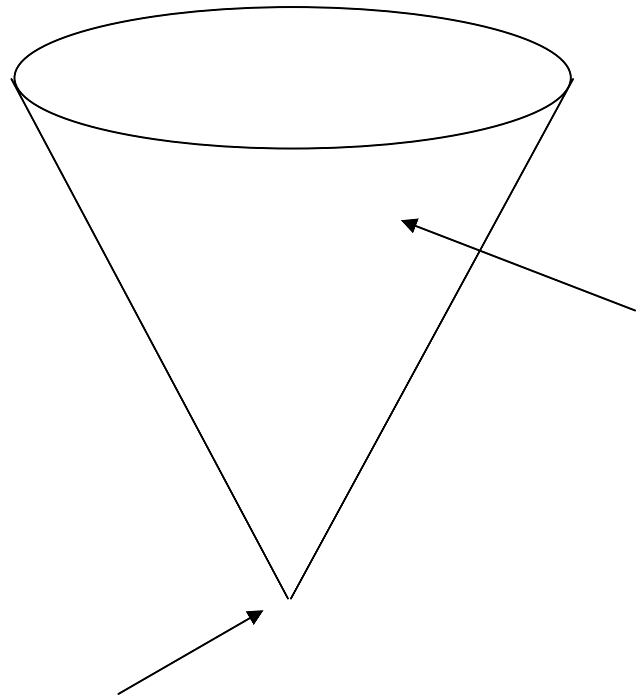
Baryons: $B = \epsilon_{a_1 \ldots a_N} Q_{i_1}^{a_1} \ldots Q_{i_N}^{a_N}$ $\tilde{B} = \epsilon_{a_1 \ldots a_N} \tilde{Q}_{i_1}^{a_1} \ldots \tilde{Q}_{i_N}^{a_N}$

These are not all independent, but satisfy the constraint

$$\det M - B\tilde{B} = 0$$

Classical Moduli Space of Vacua

$$\det M - B\tilde{B} = 0$$



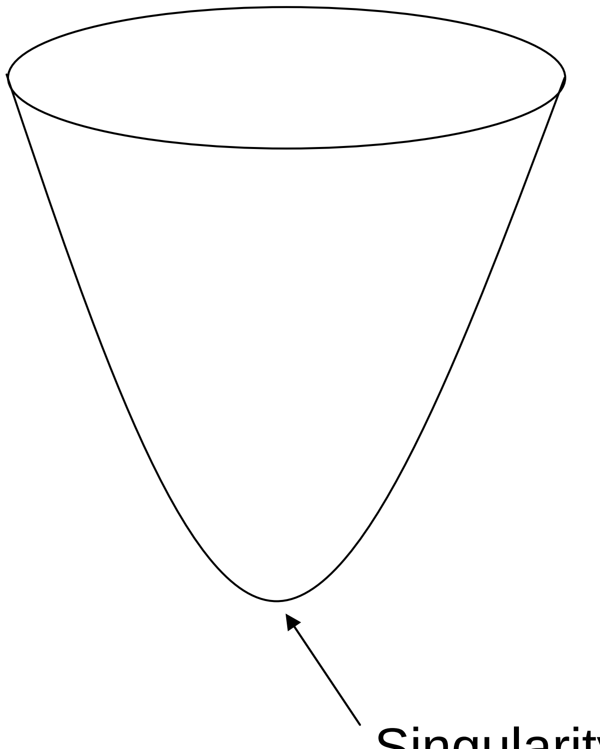
At a smooth point, we have N^2+1 massless particles

Singular point at $B = \tilde{B} = 0$ and $\mathrm{rank}(M) < N - 2$, the symmetry breaking is less than maximal $\Longrightarrow$ new massless gluons

Quantum Moduli Space

Seiberg, '94

$$\det M - B\tilde{B} = \Lambda^{2N}$$



Singularity is resolved, reflecting confinement and the fact that gluons get a mass

What Does This Mean for Vortices?

Gauge $U(1)_B$ and introduce FI parameter $v \ll \Lambda$

$\Longrightarrow$ extra D-term constraint $|B|^2 - |\tilde{B}|^2 = v^2$

BPS Vortex Equations in this theory are $F_{12}^B = e^2(|B|^2 - |\tilde{B}|^2 - v^2)$

$$\mathcal{D}_z B = \mathcal{D}_z \tilde{B} = 0$$

Key Question: When do these equations have solutions?

Key Answer: When $\tilde{B} = 0$

Classically, BPS vortices exist when $\det M = 0$
Quantum mechanically, BPS vortices exist when $\det M = \Lambda^{2N}$

Susy Breaking on the Worldsheet

We want to reproduce these effects from the $\mathbf{CP}^{N-1}$ sigma model, valid in the regime $v \gg \Lambda$

<u>**Claim**</u> The (0,2) $\mathbf{CP}^{N-1}$ model breaks supersymmetry by giving a vev to the auxiliary D-term.

<u>**Proof**</u> Work at large N. Integrate out ϕ_i to get effective potential for D, which is minimized at

$$D = \Lambda_{2d}^2$$

Susy Restoration on the Worldsheet

We need to understand how turning on meson vevs M affects the vortex

Claim $$\delta L_{\text{vortex}} = \bar{\phi}_i \frac{M^\dagger_{ij} M_{jk}}{v^2} \phi_k + \bar{\psi}_{i-} \frac{M_{ij}}{v} \psi_{j+}$$

Justification: When the meson vev is turned on, classically we must have $\tilde{Q} \neq 0$. But vortex equations $\mathcal{D}_z Q_i = \mathcal{D}_z \tilde{Q}_i = 0$ have solutions only when $\tilde{Q}$ is not sourced. This means the vortex must lie in a part of the gauge group away from $\tilde{Q}$, reducing the available moduli.

Repeat quantum calculation to find: $\det \left(\frac{M^\dagger M}{v^2} + D \right) = \Lambda_{2d}^{2N}$

The quantum theory has susy ground state when $\det M = v^N \Lambda_{2d}^N = \Lambda_{4d}^{2N}$
This agrees with Seiberg's deformation.

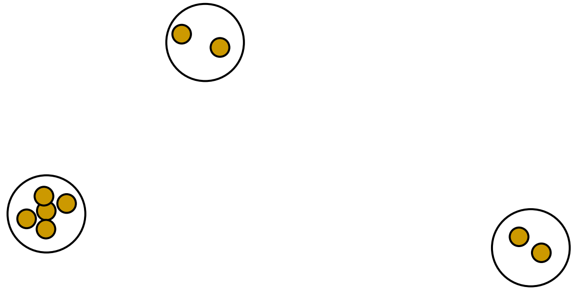
Summary

4d Quantum Deformation of Moduli Space $\Longleftrightarrow$ 2d Susy Breaking and Susy Restoration

We can also see qualitative agreement between other non-BPS quantities……like the spectrum

Why Would the Spectra Agree?

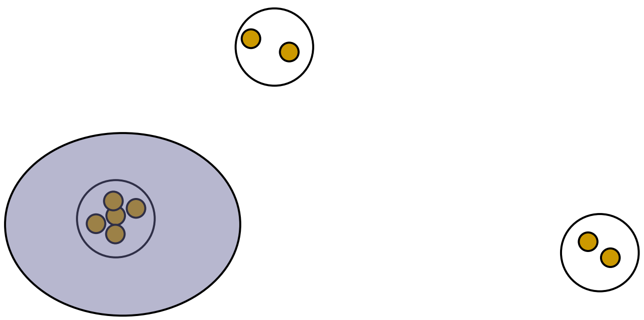
Start in the strongly coupled 4d theory with $v^2 = 0$
Spectrum = mesons and baryons



Why Would the Spectra Agree?

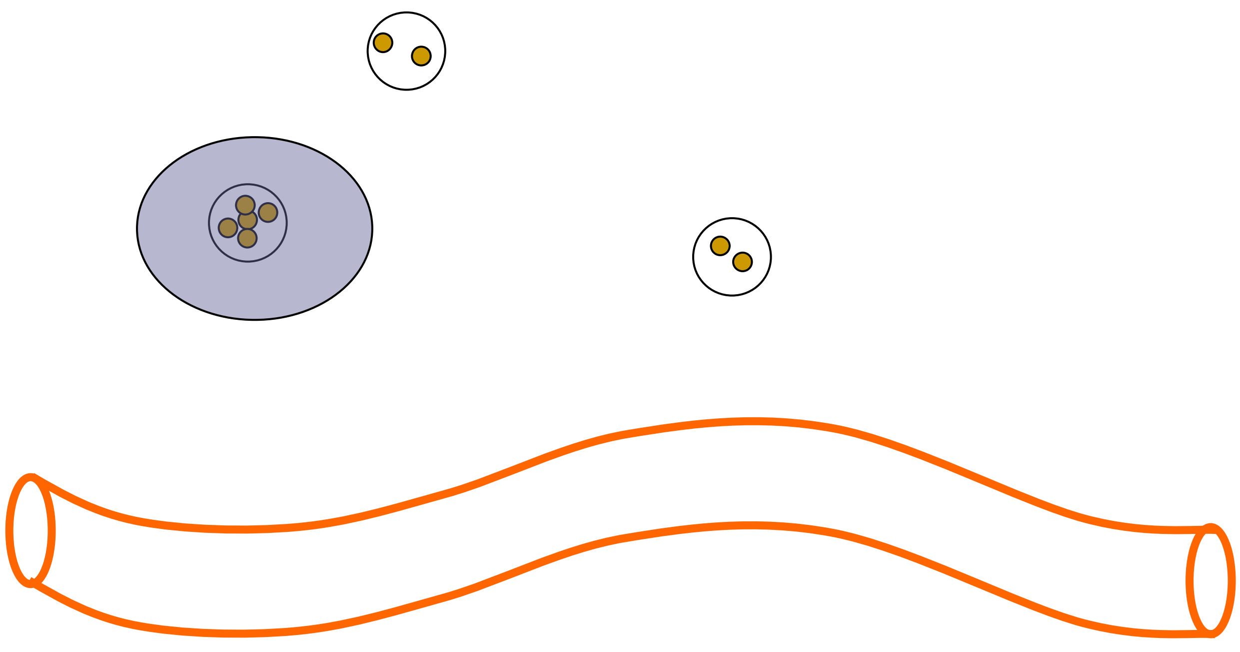
Gauge $U(1)_B$ and Higgs at scale $v \ll \Lambda$

Baryons are screened; mesons left largely unaffected.



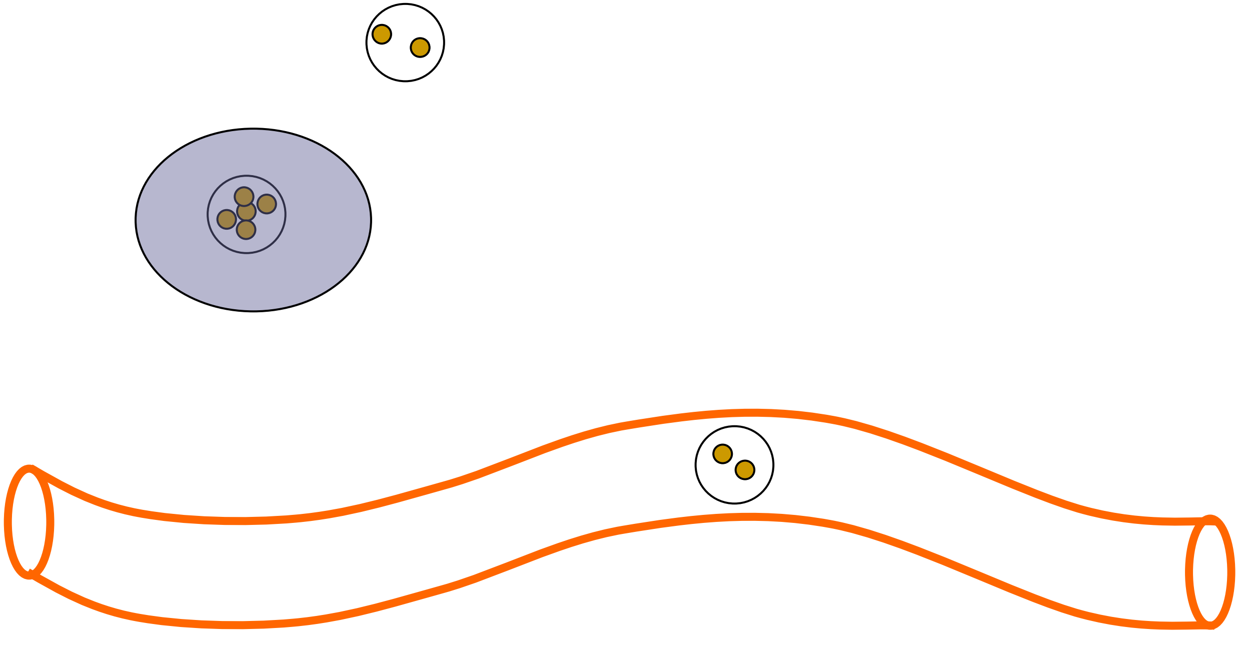
Why Would the Spectra Agree?

Introduce a vortex string. Some of the meson will form bound states with the string.



Why Would the Spectra Agree?

Now increase the ratio v/Λ . Those bound states which remain light (i.e. of order Λ) must show up as internal excitations of the 2d sigma-model.



Spectrum of the Bosonic Sigma Model

Witten, '78

Write the $\mathbf{CP}^{N-1}$ sigma model using an auxiliary U(1) gauge field.

$$L_{\rm vortex} = \sum_{i=1}^N |\mathcal{D}\phi_i|^2 - D\left(\sum_i |\psi_i|^2 - r\right)$$

$$\sum_i |\phi_i|^2 = r \ \mathrm{mod}\ \phi_i \rightarrow e^{i\alpha}\psi_i \quad \Longrightarrow \quad \mathbf{CP}^{N-1}$$

Integrate out ϕ to generate a kinetic term for the gauge field: $\frac{1}{\Lambda^2_{2d}} F^2_{01}$

This gives rise to a Coulomb force between ϕ particles

But in 2d, the Coulomb force is linearly confining. The physical states transform in the singlet and adjoint representation of the SU(N) symmetry.

Spectrum of the N=(2,2) Sigma Model

The story is the same as the bosonic case. There is a linearly confining Coulomb force, but the fermions drastically change the spectrum. The U(1) R-symmetry

$$\psi_{\pm i} \rightarrow e^{i\alpha}\psi_{\pm i}$$

is broken to Z_{2N} by an anomaly in the quantum theory. Further, the condensate

$$\langle \bar{\psi}_{+i}\psi_{-i}\rangle \sim \Lambda_{2d}^2$$

spontaneously breaks Z_{2N} to Z_2 , ensuring N isolated vacua.

The kinks interpolating between these vacua transform in the fundamental representation N of SU(N): the theory no longer confines. This is important in matching spectrum to 4d N=2 theory.

Spectrum of the N=(0,2) Sigma Model

Do we get a condensate? Might think so, but in fact it is forbidden by the Coleman-Mermin-Wagner theorem. The condensate would break the $SU(N)_L \times SU(N)_R$ chiral symmetry of the theory.

This means there is a single vacuum, and the theory consists of singlet, adjoint and bi-fundamental reps of $SU(N)_L \times SU(N)_R$

However, the 1/N expansion does predict a condensate. Resolution to this was given by Witten (in the context of the Thirring model) 30 years ago: the theory lies in the Kosterlitz-Thouless phase

$$\langle \bar{\psi}_{i+}\psi_{-i}(x)\ \bar{\psi}_{i+}\psi_{-i}(0)\rangle \sim \Lambda_{2d}^4/x^{1/N}$$

There are massless particles, transforming in the bi-fundamental representation, which give rise to this long-range correlation.

(see also Fukawa et al., hep-th/0609049)

Comparison to 4d Spectrum

- 4d Theory has chiral superfields Q and $\tilde{Q}$, in (anti)-fundamental representation of SU(N) gauge group.

- Physical spectrum consists of singlets under the gauge group, and various multiplets under $SU(N)_L \times SU(N)_R$ flavor group.

  - Meson Spectrum:
    - Massless bi-fundamental: $\tilde{Q}_i Q_j$
    - Massive singlet and adjoint $Q_i^\dagger Q_j$ and $\tilde{Q}_i \tilde{Q}_j^\dagger$

    In agreement with the spectrum of the vortex theory

  - Baryon Spectrum: Slew of tensor reps under flavor symmetry. Not seen in the vortex theory.

Summary and Future Directions

- Quantitative agreement between 2d sigma models and 4d gauge dynamics
  - N=2 Gauge Theories = N=(2,2) sigma models
    - Exact agreement between BPS mass spectra
    - Agreement between superconformal theories
  - N=1 Gauge Theories = N=(0,2) sigma models
    - Baryon vevs = worldsheet supersymmetry breaking
    - qualitative agreement between spectra
- Open Questions: N=1 Gauge Theories with $N_f > N_c$.
  - Conformal Window? Seiberg Duality?