The Innards of an Instanton

David Tong

Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Cambridge, CB3 0WA, UK
E-mail: d.tong@damtp.cam.ac.uk

Instantons are composed of partons.

1. Introduction

It is both an honour and a great pleasure to deliver this talk as part of the celebrations for Misha Shifman's 60th birthday. As a graduate student, I learned about instantons and supersymmetry from Misha's beautifully written reviews. This therefore seems like the perfect place to revisit an old idea related to instantons, but with a slightly novel supersymmetric twist.[1]

The old idea that I would like to talk about is that instantons can be thought of as containing constituent objects.[2] This idea has been mooted both for instantons in Yang-Mills theories and instantons in sigma-models. In both cases, the first hint at the existence of underlying constituents comes from simply counting the number of collective coordinates:

- Instantons in Yang-Mills theories. In $SU(N)$ Yang-Mills theory, the instanton has four translational modes, a single scaling mode, and $4N-5$ orientation modes. This gives a total of $4N$ collective coordinates.
- Instantons in sigma models. In the $\mathbf{CP}^{N-1}$ sigma-model, the instanton has 2 translation modes, a single scaling mode, and $2N-3$ orientation modes. This gives a total of $2N$ collective coordinates.

For each of these solutions the collective coordinates are Goldstone modes, arising from the action of symmetries on an instanton. However, the numerology suggests that there may be a different interpretation of the collective coordinates: as the positions of N partons which comprise the instanton.

The conjecture that instantons should be thought of as containing N partonic objects is usually framed in the context of $d = 3+1$ dimensional Yang-Mills theories and $d = 1+1$ dimensional sigma-models. In both these cases, the physics is strongly coupled in the infra-red. The hope is that in the partons — which are objects localized in (Euclidean) spacetime — are somehow liberated and the vacuum is best thought of as a correlated soup of these objects. This is then invoked to explain low-energy phenomena such as confinement or chiral symmetry breaking. Interesting ideas along these lines were presented by Zhitnitsky at the Shifmania conference and a good review can be found in.[3]

In this talk, I would like to examine the parton conjecture in the context of $d = 4+1$ dimensional Yang-Mills theories and $d = 2+1$ dimensional sigma-models. These theories are now weakly coupled in the infra-red, but strongly coupled in the ultra-violet. They are to be thought of as non-renormalizable effective field theories which require a UV completion. Moreover, the instanton solutions are now particle-like solitons, carrying finite energy as opposed to finite action. The parton conjecture becomes slightly better defined: it is the idea that the instanton should be thought of as a multi-particle state. Is this the right interpretation of the instanton in these models? And, if so, how can we tell?

2. Five-Dimensional Yang-Mills

It is known from the work of Seiberg and others that there exists a UV completion of certain Yang-Mills theories in $d = 4+1$ dimensions, at least when endowed with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry.[4] However, we don't know a whole lot about the details of this UV theory.

The story is particularly interesting in the case of $\mathcal{N} = 2$ supersymmetry. (This means 16 supercharges). Here it is known that the UV completion of $d = 4+1$ dimensional $SU(N)$ Yang-Mills is really a conformal field theory in $d = 5+1$ dimensions, usually referred to as the $(2,0)$ theory.[5] This theory is compactified on a circle of radius R, which is related to the 5d Yang-Mills coupling by $R = e^2/8\pi^2$. Little is known about the dynamics of the $(2,0)$ theory and, in particular, there is no known Lagrangian formulation. What little knowledge we do have comes from the gauge-gravity correspondence. Most strikingly, it can be shown that the $(2,0)$ theory has $\sim N^3$ degrees of freedom.[6] This is many more than the $\sim N^2$ degrees of freedom that are seen in the infra-red Yang-Mills theory. This means that if we're looking for the UV completion of the theory, then we need to find a lot of states! Understanding how these N^3 degrees of freedom arise is likely to tell us something important about the degrees of freedom of M-theory.

The instanton in $d = 4 + 1$ dimensional Yang-Mills plays an important role in this story: it is the Kaluza-Klein mode coming from the theory in $d = 5+1$ dimensions. This can be seen already in the classical mass formula,

$$M_{\rm inst} = \frac{8\pi^2}{e^2} = \frac{1}{R} \tag{1}$$

The proposal that I would like to explore is the following: *The instanton in $d = 4 + 1$ dimensional Yang-Mills should be thought of as an N particle state*. Moreover, the N partons are to be thought of as the remnant of the UV degrees of freedom which comprise the $(2, 0)$ theory.

Let's look at some circumstantial evidence for this proposal. Firstly, we can compute the free energy in the infra-red and ultra-violet and ask: at what energy does the scaling change from $\sim N^2$ to $\sim N^3$? We can attack this calculation using the supergravity dual,[7] but the correct result also follows simply from assuming a second order phase transition and equating the two free energies at the critical point,

$$F \sim N^2T^5 \sim RN^3T^6 \tag{2}$$

from which we learn that the cross-over happens at the critical temperature

$$T \sim \frac{1}{NR} \sim \frac{1}{e^2N} \tag{3}$$

The factor of $1/N$ is all-important here. It tells us that, whatever the new degrees of freedom are, they come in at the energy scale below the Kaluza-Klein scale. Indeed, this had to be the case: the 5d theory becomes strongly coupled at the energy scale $E \sim 1/e^2N$ and it is here that new UV degrees of freedom are required to render the theory well-defined in the UV. This is $1/N^{\rm th}$ the mass of an instanton. These are our conjectured partons.

There is more circumstantial evidence for the partonic nature of the instanton. There is a refinement in the count of the number of degrees of freedom that comes from looking at the anomaly coefficient. For $G = ADE$ gauge group, the coefficient is,[8,9]

$$c_2(G) \times |G| \tag{4}$$

where $c_2(G)$ is the dual Coxeter number (equal to $c_2(G) = N$ for $G = SU(N)$) while $|G|$ is the dimension of the gauge group (equal to $|G| = N(N-1)$ for $G = SU(N)$). For $G = SU(N)$, this reproduces the leading order scaling, but also gives the subleading contribution: $N^3 - N$. However, for arbitrary $G = ADE$ gauge group, this formula has an interesting interpretation in terms of the partonic nature of instantons for $G = ADE$ gauge group. This is because the dimension of the instanton moduli space is known to be $4c_2(G)$. If the anomaly coefficient is a good measure of the number degrees of freedom, it is suggesting that each parton itself transforms in the adjoint of the gauge group G. Making sense of this statement would go a long way towards understanding the parton picture of instantons.

Finally, the partonic interpretation of instantons naturally resolves a puzzle that arises upon quantization: the scaling mode of the instanton gives rise to a continuous energy spectrum above $M_{\rm inst}$. This is very peculiar behaviour for a one-particle state in quantum field theory. However, it is entirely natural for a multi-particle state.

The upshot of the above discussion is that the partonic interpretation of instantons would explain several known features of the UV behaviour of 5d Yang-Mills theories. However, there are lots of questions that remain unanswered. Most pertinent among these is: why are the partons confined to live within the instanton? This is not confinement as we know it in QCD. The existence of the instanton moduli space means that the partons are individually free to wander $\mathbf{R}^4$. Nonetheless, there are interactions between them which mean that are denied an existence on their own. What is the cause of these interactions? And how can we understand the nature of the partons given that we only have access to the low-energy physics of Yang-Mills theories.

I have not been able to answer these questions in the context of Yang-Mills. Rather, in the rest of this talk I will retreat and discuss a simple toy model which contains many of the same problems, but in a context where we can completely understand the relevant physics. In fact, the toy-model is very familiar to everyone who got stuck with QCD and looked for inspiration in something simpler: it is the $\mathbf{CP}^{N-1}$ sigma-model. Since we are interested in an analogy for Yang-Mills theory in $d = 4 + 1$ dimensions, we will look at the sigma-model in $d = 2 + 1$ dimensions.

3. Three Dimensional Sigma Models

The toy model that we consider in a supersymmetric sigma model in $d = 2+1$ dimensions. Specifically, we will consider $\mathcal{N} = 4$ supersymmetry (which means eight supercharges). While the low-energy physics will be a sigma-model, in contrast to the case of Yang-Mills theory, we will also stipulate the UV completion: it is a gauge theory first discussed by Intriligator and Seiberg.[10] We first describe this gauge theory and then explain how the sigma model emerges in the infra-red. We construct the gauge theory out of vector multiplets and hypermultiplets. These contain the following fields:

- Vector multiplet: $V = (A_\mu, \phi_i, \text{fermions})$, ϕ_i are real scalars with $i = 1, 2, 3$.
- Hypermultiplet: $Q = (q, \tilde{q}, \text{fermions})$. q and $\tilde{q}$ are both complex scalars.

The gauge theory that we consider is $U(1)^N$ with N hypermultiplets. The charges of the matter multiplets are best summarized by the following quiver diagram in which each node corresponds to a gauge group and each link to a hypermultiplet.

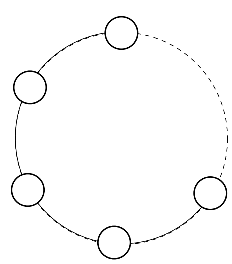

Fig. 1. The quiver diagram for the UV gauge theory.

We will need to introduce a couple of parameters for this theory. Each gauge group is assigned coupling constant g^2. Each matter multiplet is assigned bare mass m. However, the physical mass, M_a, $a = 1, \ldots, N$ of the N^{th} matter multiplet also depends on the expectation value of the scalars in the vector multiplets in the a^{th} and $(a+1)^{\text{th}}$ gauge group[a],

$$M^a = \phi^a - \phi^{a+1} + m \tag{5}$$

While the quiver gauge theory happily describes the UV physics, our real interest is in the infra-red. At low energies, we integrate out the massive hypermultiplets and seek an effective theory describing the massless vector multiplets: these are $3N$ scalar fields ϕ_i^a together with $3N$ photons, each of which can be dualised in favour of a periodic scalar field,

$$F_{\mu\nu}^a \sim g^2 \epsilon_{\mu\nu\rho} \partial_\rho \sigma^a \tag{6}$$

Integrating out the hypermultiplets induces derivative interactions between the vector multiplet fields. The resulting physics is a low-energy sigma-model on the Coulomb branch with target space[10]

$$\mathbf{R}^3 \times \mathbf{S}^1 \times T^\star \mathbf{CP}^{N-1} \tag{7}$$

The fields ϕ_i^a and σ^a provide coordinates on this space. We won't go through the derivation of this sigma-model, but it will be useful to sketch how it works in the simple example of $N = 2$. In this case, the quiver has just two nodes. The diagonal $U(1)$ gauge field has nothing charged under it and decouples (it will give rise to the $\mathbf{R}^3 \times \mathbf{S}^1$ factor of the target space). Meanwhile, the axial $U(1)$ couples to two hypermultiplets with charge $+1$ and -1. The discussion is simplest if we ignore two of the scalars in the vector multiplet (these will give rise to the cotangent bundle $T^\star$) and focus just on a single scalar ϕ and the dual photon σ. The low-energy dynamics for these fields is given by

$$\mathcal{L} = \frac{1}{g_{\text{eff}}^2} (\partial\phi)^2 + g_{\text{eff}}^2 (\partial\sigma)^2 \tag{8}$$

where the effective coupling constant gets one-loop contributions from each of the hypermultiplets,

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2} + \frac{1}{m-\phi} + \frac{1}{m+\phi} \tag{9}$$

The target space for $|\phi| < |m|$ is drawn in the figure below. Topologically, it is a sphere $\mathbf{CP}^1$. For finite g^2, the space has the shape of a rugby ball: it has a squashed metric with only $U(1)$ isometry. In the limit $g^2 \to \infty$,

[a]This expression actually describes a triplet of masses for each hypermultplet, corresponding to the triplet of scalars in a vector multiplet. For details of how this works, and why it's not important for the following discussion, see.[1]

the target spcae becomes round, like a football[b], and the isometry of the metric is enhanced to $SU(2)$.

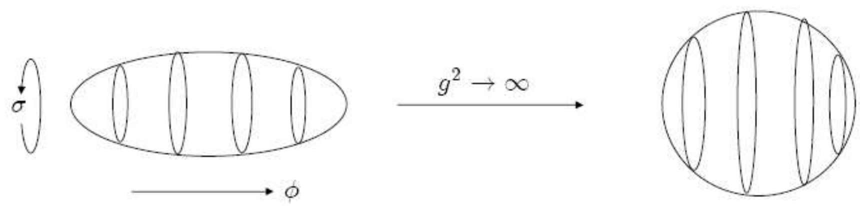

Fig. 2. The target space for the low-energy sigma model.

The Soliton

The low-energy sigma-model has a soliton. It is the sigma-model lump, a.k.a. the sigma-model instanton that we discussed in the introduction. In $d = 2 + 1$ dimensions, this is a particle-like state. For target space $\mathbf{CP}^1$ (either with a squashed or round metric), the spatial variation of the fields is governed by the first-order Bogomolnyi equation

$$\partial_\mu \phi = g^2_{\rm eff} \epsilon_{\mu\nu} \partial_\nu \sigma \tag{10}$$

Similar first order equations hold for the soliton in $\mathbf{CP}^{N-1}$.

What does this soliton correspond to in the microscopic theory? There are a few clues. Firstly, it is a BPS state. Secondly, its mass is Nm. In fact, this is enough to identify the microscopic origin of this soliton: it is an N-particle state constructed from taking a hypermultiplet field from each link of the quiver ring:[11] $Q_1 Q_2 \ldots Q_N$. Now we see the importance of this three-dimensional example. It is an explicit model in which the partonic nature of the soliton is realised. A single soliton in the low-energy theory is indeed interpreted as a multi-particle state in the UV. Our goal now is to ask the question in reverse: given only access to the IR physics, what can we learn about the UV physics by studying the soliton? Of course, hindsight is a wonderful thing and we intend to employ it to its full extent in our study. Nonetheless, the answer is rather surprising. By studying the properties of the soliton, we will be able to reconstruct the full UV physics. In particular,

[b]This sporting analogy is to be taken in the European sense.

this means resolving the quantum numbers of the partons that lurk inside the soliton. In the rest of this talk, we will see how this comes about.

The Partons

Let's first ask how we can see the partons inside the soliton in this model more explicitly. As we mentioned in the introduction, the counting of collective coordinates is certainly suggestive of an interpretation in terms of the positions of N objects on the plane. However, if we simply plot the energy configuration of a single soliton, it just looks like a round blob with no hint of any internal structure. How do we reconcile these statements?

In fact, the UV theory is already telling us the right place to look. Recall that the round Fubini-Study metric on $\mathbf{CP}^{N-1}$ only arises in the limit $g^2 \to \infty$. If, instead, we study the sigma-model at finite g^2, then the metric is squashed with only a $U(1)^{N-1}$ isometry. The full $2N$ collective coordinates of a single soliton survive at finite g^2 (they are protected by index theorems), but they are no longer Goldstone modes. They now explicitly determine the positions of the partons. For example, the figure below shows the energy profile for a single soliton for a $\mathbf{CP}^1$ target space. We see that the two partons dramatically reveal themselves as the target space is squashed.

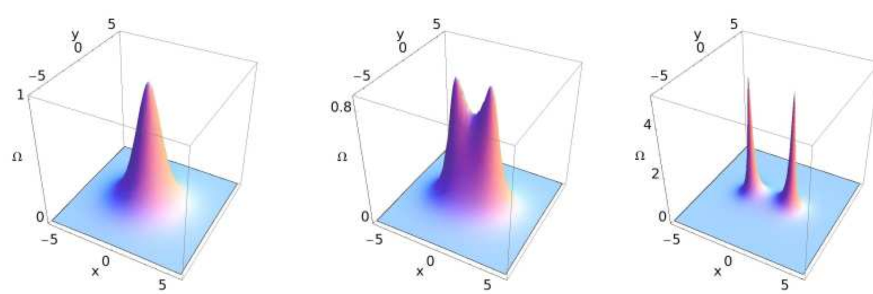

Fig. 3. The two partons inside a $\mathbf{CP}^1$ lump for $m/g^2 = 0$, 1 and 2.

A similar phenomenon happens for target space $\mathbf{CP}^{N-1}$. Here is the example of $\mathbf{CP}^2$, where the single soliton decomposes into three partons. A similar mechanism to see the partons was described in the talk at Shifmania by Ken Konishi.[12]

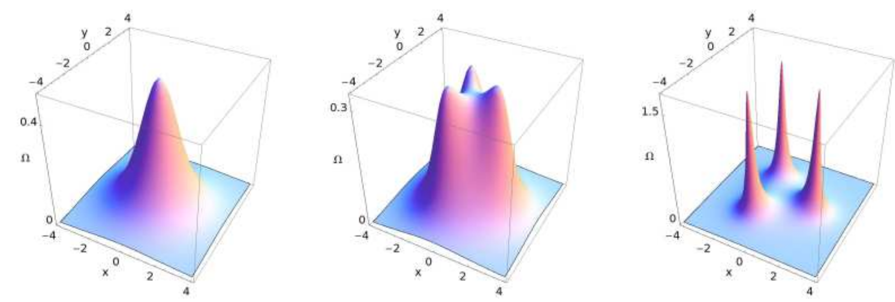

Fig. 4. The three partons inside a $\mathbf{CP}^2$ lump as the target space is squashed.

We can also see explicitly that the collective coordinates change the positions of the partons. For example, we could keep the “scale size” fixed and change the “orientation” modes of the soliton. For a round target space, this would leave the energy profile unchanged. However, with the squashed target space, the orientation modes govern the relative positions of the partons. For example, here’s some plots showing the single lump in $\mathbf{CP}^2$ with different values of the collective coordinates.

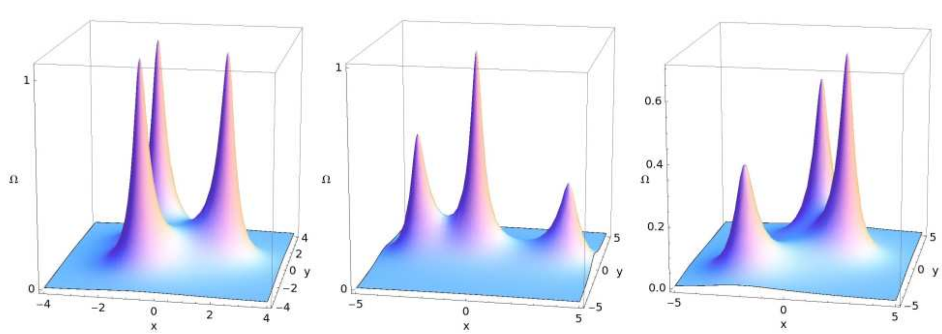

Fig. 5. Moving the partons inside a $\mathbf{CP}^2$ lump.

Let’s now turn to the question of confinement. Why must the partons sit together inside a lump even though they are free to roam? The answer from the microscopic theory is simple: any electrically charged state gives rise to an electric field which asymptotically goes like $E \sim 1/r$. This gives rise to a logarithmically divergent contribution to the energy. Only gauge singlet states have finite mass, and this is the reason that the partons are bound together in the state $Q_1, \ldots Q_N$. Moreover, this log-divergence re-appears in the low-energy effective theory once we ask the partons to move: it is seen as a log-divergence in the metric on the soliton moduli space.

We have seen above that squashing the sigma-model allows us to graphically see the partons that sit inside the instanton. But how do we see their quantum numbers? Here we explain this for solitons in $\mathbf{CP}^1$. The Bogomolnyi equation for the soliton is given in (10). The k soliton has a moduli space of solutions of dimension $2k$. Using the duality transformation (6), we can rewrite this equation as

$$F_{0\mu} = \partial_\mu \phi \tag{11}$$

These are the dual Bogomolnyi equations. In contrast to (10), they have *no* smooth solutions. But this is entirely expected: after a duality transformation, solitons become fundamental excitations. These should not be associated to smooth solutions, but rather to solutions of equations with sources.

$$\partial_\mu \left(\frac{1}{g_{\rm eff}^2} F_{0\mu} \right) = \sum_{n=1}^{k} \delta(z - z_n^+) - \delta(z - z_n^-) \tag{12}$$

One can show that solutions to (12) and (11) coincide with smooth solutions to (10). The positions of the sources $z_n^\pm$ become coordinates on the moduli space of the soliton. This provide a very simple and explicit map between fundamental excitations and solitons in a field theory.

The construction of the dual Bogomolnyi equation also works for solitons in $\mathbf{CP}^{N-1}$. It allows us to determine the quantum numbers of the partons. In this way, we can reconstruct the quiver diagram of the UV theory, details which one might have reasonably expected were lost to the winds of the renormalization group by the time we restriced ourselves to the low-energy physics. Details of this can be found in the longer paper.[1]

4. Summary: Questions, not Answers

Our toy model in $d = 2 + 1$ dimensions did all that we hoped. It provides an explicit setting where the single soliton has the interpretation of an N-particle state. The partons inside the soliton are identified with the degrees of freedom necessary to form a UV completion of the sigma-model. Moreover, by studying the properties of this soliton we were able to reconstruct the quantum numbers of these partons and therefore the UV physics.

All of this is heartening. However, the real question remains: can we do the same for Yang-Mills instantons in $d = 4 + 1$ dimensions? What is the confinement mechanism that keeps these partons trapped inside an instanton and what is this telling us about the microscopic dynamics of the $(2, 0)$ theory? I don't yet have answers to these questions but I hope that further study of the Yang-Mills instanton will provide some vital clues.

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