

# Physics and the Integers

David Tong

Department of Applied Mathematics and Theoretical Physics  
University of Cambridge, UK  
d.tong@damtp.cam.ac.uk

**Abstract:** I review how discrete structures, embodied in the integers, appear in the laws of physics, from quantum mechanics to statistical mechanics to the Standard Model. I argue that the integers are emergent. If we are looking to build the future laws of physics, discrete mathematics is no better a starting point than the rules of scrabble.

## A History Lesson

*“God made the integers, the rest is the work of man” — Leopold Kronecker [1]*

I have never really understood this quote. It may be fine for mathematicians, but it doesn't seem to gel with how I understand the laws of physics. In part, the purpose of this essay is to explain why.

I recently learned that it's not just me who disagrees with Kronecker. At the time, *everyone* disagreed with him! This quote is part of a polemic by Kronecker against developments in mathematics in the late 1800s such as irrational numbers, Cantor's set theory and the Bolzano-Weierstrass theorem. Kronecker, an old distinguished mathematician, thought these new developments undignified. He preferred his mathematics constructive and discrete. Needless to say, it did not make Kronecker a popular man among his peers.

More than a century later, no mathematician would deny the importance and utility of the developments that Kronecker railed against. Yet, an informal survey among my colleagues suggests that many harbour some sympathy for his statement. The integers hold a special place in the heart of mathematics. Many of the most famous unsolved conjectures relate to the properties of the primes. More importantly, the integers are where we start mathematics: they are how we count.

At the same time that mathematicians were arguing about the building blocks of numbers, a parallel debate was taking place among physicists concerned with the building blocks of matter. Although the ancient Greeks discussed the possibility of atoms, it wasn't until the 1800s that they became a useful concept. The development of kinetic theory by Maxwell, Boltzmann and others showed that the laws of thermodynamics and gases could be derived from Newtonian physics combined with a healthy dose of statistics. But these arguments were not convincing to everyone. The kinetic theorists could not offer a “constructive proof” of the existence of atoms. The evidence was only indirect [2].

While some of the objections to atoms were philosophical, most notably from Mach, there was also strong opposition among physicists and chemists. In 1882, Planck proclaimed “atomic theory, despite its great successes, will ultimately have to be abandoned in favor of the assumption of continuous matter” [3]. In part, the reluctance of physicists to accept atoms sprung from an unwillingness to retreat to probabilistic reasoning — an issue that was to reappear even more acutely some decades later. However, there was also a second argument against atoms, stated most robustly by Ostwald: Unification. The laws of thermodynamics refer only to continuous quantities such as energy. Similarly, Maxwell's theory of electromagnetism describes the motion of continuous electric and magnetic fields. The writing on the wall in the mid-to-late 1800s was that physics was moving away from the discrete particles postulated by Newton. The fashion was for the continuous. How wrong they were....

## **The Integers in Nature**

It is not, a priori, obvious that the integers have any role to play in physics. Place yourself in the shoes of the nineteenth century physicist, struggling to understand electricity, magnetism, heat and fluid mechanics. Where is the discrete? Try counting the ripples on a pond or the wisps of steam emitted by a kettle. Would you just include the big ones? Or admit that these are not really objects that can be placed in one-to-one correspondence with the integers.

Of course, not everything in Nature is ripple or wisp. There are seemingly discrete objects. The number of people reading this article will, to good approximation, be integer. Yet even in situations where it naively seems obvious that integers are important, a closer examination often reveals that this is illusory. For example, I was told at school that there are 9 planets in the solar system. Now there are 8. Or maybe 13. Or, to give the honest answer: it depends. The Kuiper belt contains objects in

size ranging from a few thousand kilometers to a few microns. You can happily count the number of planets but only if you first make a fairly arbitrary distinction between what is a planet and what is just a lump of rock. Ultimately, counting planets is not very different from counting ripples on a pond: if you want to do it, you've got to just include the big ones.

As this example highlights, the counting that is evident in mathematics is not so easy in the real world. To find the integers in physics, we need Nature to provide us with objects which are truly discrete.

Fortunately, such objects exist. While the definition of a planet may be arbitrary, the definition of an atom, or an elementary particle, is not. Historically, the first place that the integers appeared was in the periodic table of elements. The integers labelling atoms – which, we now know, count the number of protons – are honest. Regardless of what developments occur in physics, I will happily take bets that we will never observe a stable element with  $\sqrt{500}$  protons that sits between Titanium and Vanadium. The integers in atomic physics are here to stay.

In fact, once we are in the atomic world, the integers are everywhere. This is what the “quantum” of quantum mechanics means. One of the first examples we learn is that the energy levels of hydrogen are given by  $E = -E_0/n^2$  where  $E_0$  is the ground state energy and  $n$  is an integer.

More subtle quantum effects can coax the integers to appear in macroscopic, even dirty, systems. The quantum Hall effect is a phenomenon that occurs when semiconductors are placed in a magnetic field at low temperatures. The Hall conductivity, which describes how current flows perpendicular to an applied electric field, is given by

$$\sigma = n \frac{e^2}{h}$$

where  $e$  is the electron charge and  $h$  is Planck's constant and, once again,  $n$  is an integer (or, if you do things more carefully, a rational number). These integers have been measured to an accuracy of one part in a billion, one of the most precise experiments in all of physics.

## The Integers are Emergent

The examples above show that discrete objects undoubtedly appear in Nature. Yet this discreteness does not form the building blocks for our best theories. Quantum mechanics is not constructed from the integers. The fundamental object is a complex-valued wavefunction  $\psi$  obeying the Schrödinger equation. Nor are there integers in the Schrödinger equation itself. For example, the Hydrogen atom is governed by

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}) - \frac{e^2}{4\pi\epsilon_0 r}\psi(x) = E\psi(\vec{x})$$

The left-hand side contains nothing discrete. The integers appear on the right-hand side only when we solve this equation. In fact, even that is not ensured. There are classes of solutions, so called scattering states, for which  $E$  takes continuous values. And there is a second class, the bound states, which require a normalization condition on  $\psi$  and are labelled by integers

$$E = -\frac{me^4}{32\pi^2\hbar^2\epsilon_0^2} \frac{1}{n^2} \quad n \in \mathbf{Z}$$

At least as far as the Hydrogen atom is concerned the lesson is clear: God did not make the integers. He made the complex numbers and the rest is the work of the Schrödinger equation. The integers are not inputs of the theory. They are outputs. The integers are an example of an emergent quantity.

The fact that the integers are emergent is highlighted even more dramatically in the example of the quantum Hall effect where they arise from the collective dynamics of a macroscopic number of particles.

Perhaps more surprisingly, the existence of atoms – or, indeed, of any elementary particle – is also not an input of our theories. Despite what we learn in high school, the basic building blocks of Nature are not discrete particles such as the electron or quark. Instead our fundamental laws of physics describe the behaviour of *fields*, continuous fluid-like objects spread throughout space. The electric and magnetic fields are familiar examples, but our best description of reality — the Standard Model — adds to these an electron field, a quark field, and several more. The objects that we call fundamental particles are not fundamental. Instead they are ripples of continuous fields, moulded into apparently discrete lumps of energy by the framework of quantum mechanics which, in this context, is called *quantum field theory*. In this way, the discreteness of the particle emerges. It is conceptually no different from the energy levels of the Hydrogen atom.

Wherever the integers arise in Nature, we are led to the same conclusion: the integers are emergent. They are no more fundamental than the concepts of temperature, smell or the rules of scrabble.

## Searching for Integers in the Laws of Physics

While the quanta of quantum mechanics are emergent, we could also play a more fundamental version of the “count-the-planets” game, this time trying to find integers within the laws of physics themselves. Are there inputs in the fundamental equations that are truly discrete? At first glance, it is obvious that there are; at closer inspection, it is more murky. I should warn the reader that this small section is substantially more technical than the rest of the paper and little of the general argument will be lost if it is skipped.

Let me get the ball rolling by offering some well-known examples of integers in the Standard Model. There are *three* generations. Each contains *two* leptons and *two* quarks. The forces are described by the gauge group  $SU(3) \times SU(2) \times U(1)$ . The representations of these gauge groups are labelled by integers. And so on and so on. Integers, integers everywhere. Or are there? All the examples above are really counting the number of particles species in the Standard Model, whether fermions or (gauge) bosons. But this is a famously difficult question to make mathematically precise in an interacting quantum field theory. The problem is that particles can mutate into each other and the boundary between them becomes blurred: a neutron can split into a proton, an electron and a neutrino, but it’s not useful to think of the neutron as containing these three other particles. Should we count this as one particle or three particles or four particles?

In fact, it is still an open problem to count the number of degrees of freedom in a general quantum field theory. But good candidate answers exist: the entropy, or, for conformal theories, the  $a$  and  $c$  coefficients of the gravitational anomaly [4]. In each case, the answer is not an integer<sup>1</sup>. The counting that gives us the integers in the Standard Model is merely an artifact of working with free fields.

Here is another example of an integer in the laws of physics: the number of observed spatial dimensions in the Universe is three. It’s a nice round number. But is it just an

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<sup>1</sup>It is worth noting that the problem of counting degrees of freedom has been solved for quantum field theories in  $d = 1 + 1$  dimensions by Zamolodchikov’s “c-theorem” [5]. At the fixed points, the degrees of freedom are counted by the central charge. Again, it is not in general integer valued.

approximation to something deeper that doesn't have to be an integer? The answer is that we don't yet know, but it's easy to cook-up scenarios in which this is the case. Indeed, mathematically the Hausdorff dimension of a set need not be integer and, at a push, can be used to count dimensions of objects in Nature: the coastline of Great Britain supposedly has a dimension somewhere around 1.3 [6].

Yet even if the Universe is not a fractal (and there is no compelling reason why it should be) studies of strongly interacting quantum systems reveal many examples where the dimension of space in which a theory lives is ambiguous. Perhaps the most striking is the AdS/CFT correspondence where quantum field theories are entirely equivalent to quantum gravity theories in higher dimensions [7]. Another example is the relationship between Calabi-Yau sigma-models and Landau-Ginsberg theories [8], where the smooth geometry of a background space can smoothly and continuously dissolve, until one is left with an interacting theory with no hint of a geometric interpretation. The dimensions of space are not the robust objects that they appear. At heart, the issue is exactly the same as that described above: the dimensions of space count the number of degrees of freedom of a particle that lives in it. And, like ripples on the pond, the number of degrees of freedom is not so easy to count in interacting theories.

Finally, I would like to finish this section with an example where I cannot think of a way to wriggle out: the laws of physics refer to *one* dimension of time. It appears to be much more challenging to replace this particular integer with anything else.

### **Could the Known Laws of Physics be Fundamentally Discrete?**

Above I have argued that the presence of discrete structures in Nature is either emergent or illusory. Either way, there is no evidence for an underlying digital reality. But absence of evidence is not evidence of absence. Is it possible, nonetheless, that Nature is at heart discrete?

It naively appears that there is no obstacle to a discrete reality. While quantum mechanics moulds discreteness from underlying continuity, there are many places in physics where continuity emerges from discreteness. Fluid mechanics offers perhaps the simplest example. The atomic constituents are almost irrelevant for the large scale description of liquids or gases. The microscopic details affect only a handful of phenomenological macroscopic parameters such as the virial coefficients in the equation of state, the viscosity and related transport coefficients.

One can go yet further. The beautiful story of universality in statistical mechanics tells us that, at a second order phase transition, even the sparse microscopic information hidden in phenomenological parameters can get washed away. This is the subject of critical phenomena. At the liquid-gas critical point, all information about the properties of the atoms is lost and one has no way of knowing if the underlying objects are water molecules or magnetic spins. This is particularly striking in the case of magnetic spins (the Ising model) where the microscopic theory lives on a lattice in space, breaking rotational invariance. Yet there is no hint of this at large distances where we see only an emergent continuum, the rotational symmetry fully restored. Could a mechanism of this type perhaps underlie the laws of physics?

Before we get ahead of ourselves and start looking into the unknown, it would seem prudent to first ask whether the laws of physics that we already have to hand are compatible with a discrete underlying structure. Such a formulation is not just of philosophical value. The equations of physics are hard and humans are not very good at solving hard equations. Computers are much better. To formulate the laws of physics in a discrete manner means to write them in such a way that they could be simulated on a computer. Can we simulate the known laws of physics on a computer?

The answer to this question is, at least to me, extremely surprising: no one knows how to formulate a discrete version of the laws of physics.

At first sight, this seems very strange. All the laws of physics that we learn as an undergraduate are presented in terms of differential equations and, while it may be difficult in practice to get reliable numerical results for certain partial differential equations, there is no problem of principle. One simply needs to replace continuous derivatives with finite differences. So, although there are very real difficulties in simulating both Einstein's equations of General Relativity and the Navier-Stokes equations of fluid dynamics, for the purposes of the present discussion I will say that we understand how to formulate discrete versions of both these theories.

The problem with finding a discrete formulation arises when we turn to quantum field theory. Here the weapon of choice is not the differential equation, but instead the path integral<sup>2</sup>. This is a functional integral, meaning that one doesn't integrate over a

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<sup>2</sup>One *could* formulate quantum field theory entirely in terms of differential equations. Now the trouble arises because the operator acts on an infinite dimensional space, the wavefunctional. Saying that quantum field theory can be written in terms of differential equations is analogous to saying that the Schrödinger equation can be written as a matrix eigenvalue equation: formally true, as long as

domain in, say, the real numbers, but instead over a domain of functions.

Mathematicians have long struggled to make sense of the path integral in all but simplest cases. The infinities that arise in performing functional integrations are beyond the limits of rigorous analysis. Of course, that hasn't stopped physicists using the path integral to great effect, both in particle physics and, ironically, in mathematics.

One of the most important tools that has been developed to study path integrals is a discretised version of the quantum field called *lattice field theory*. Here the spacetime continuum is replaced with a discrete lattice structure. When the parameters of the underlying field theory are tuned to a critical point, the details of the lattice wash out and, if we only look at large distances, the theory appears once again continuous with translation and Lorentz invariance intact. This provides the paradigmatic example of how seemingly continuous physics can emerge from an underlying discrete structure.

However, it is not always so easy to construct a lattice version of a quantum field theory. The trouble lies with fermions, objects which carry half-integer spin so you have to turn around twice before you get back to where you started. There is a long history of headaches associated with lattice fermions, many of them enshrined in the celebrated “no-go theorem” due to Nielsen and Ninomiya [9]. Important progress in the 1990s [10] showed how one can circumvent many, but not all, of these problems. The current state of the art is that there is just a single class of quantum field theories which physicists do not know how to simulate on a computer [11]. This is the class in which fermions that spin in an anti-clockwise direction experience different (non-Abelian) forces from those that spin in a clockwise direction. Such theories are referred to as *chiral*.

Chiral theories are interesting and delicate. Subtle effects known as anomalies are always lurking, threatening to render the theory mathematically inconsistent. For this reason chiral quantum field theories are rather special. But perhaps the most special among them is the Standard Model. This is a chiral theory because only fermions that spin anti-clockwise experience the Weak force. Chirality is one of the most striking and important features of the Standard Model. Yet, when it comes to constructing a lattice version of the theory, it has consequence: no one knows how to write down a discrete version of the Standard Model. Which means that no one knows how to write down a discrete version of the current laws of physics.

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you don't care too much about taking limits, but practically useless.



## Could the Unknown Laws of Physics be Fundamentally Discrete?

Until now, I have tried to resist speculation, preferring to mostly restrict attention to a study of the known, confirmed laws of physics. But what does this tell us about the underlying reality of the world? Are the true laws of physics discrete? Is spacetime discrete? Should we all be searching for the green digital rain seen by Neo at the end of the *The Matrix*?

The first piece of evidence offered in this essay is that the integers are emergent. I find it striking that the discreteness we see in the world is not sewn into the equations of physics, but arises only when we solve them. It didn't have to be this way. Indeed, the very first attempts to write down a quantum theory placed the discreteness at the heart of the equations. This is Bohr-Sommerfeld quantization: one takes a classical phase space and parcels it up into little pieces of area  $\hbar$ . It is often a good first approximation to understanding a system. But, ultimately, it is not correct. Nature found a more interesting way: discreteness in the world is simply the Fourier transform of compactness.

If the history of physics has taught us one thing, it is that we should be careful in extrapolating results beyond their realm of applicability. But I'm going to do it anyway. I find it hard to believe that having found one beautiful way to implement discreteness, Nature would shrug her shoulders at the next level and simply throw it in by hand. She is surely more subtle than that.

It is not entirely clear what to make of the second piece of evidence presented in this essay: our inability to simulate the Standard Model on a computer. It is difficult to draw strong conclusions from a failure to solve a problem and the huge progress made in this area a decade ago suggests that it is probably just a very difficult issue waiting to be solved with fairly conventional techniques. I included it in this essay mostly as a warning shot to those who would insist that it is obvious that Nature is digital. But it may be worth considering the possibility that the difficulty in placing chiral fermions on the lattice is telling us something important: the laws of physics are not, at heart, discrete. We are not living inside a computer simulation.

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