A Review of Solitons in Gauge Theories

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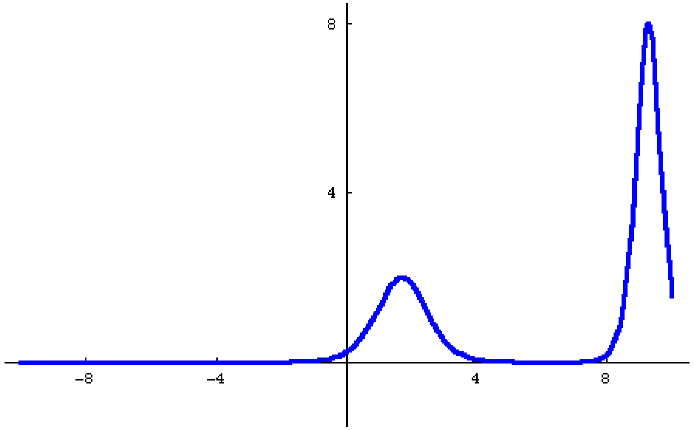
Introduction to Solitons

- Solitons are particle-like excitations in field theories
- Their existence often follows from general considerations of topology and symmetry

An example: the KdV equation

$$u_t - \frac{3}{2} u \, u_x - \frac{1}{4} u_{xxx} = 0$$

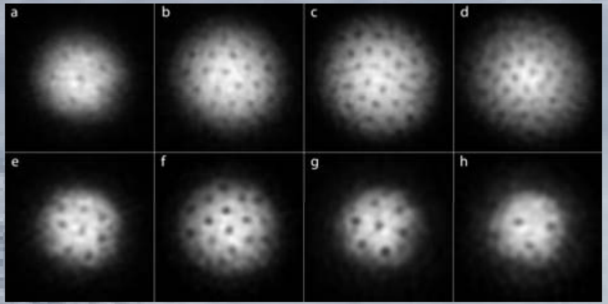


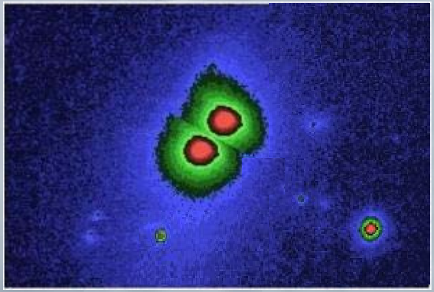
Animation by A. Kasman

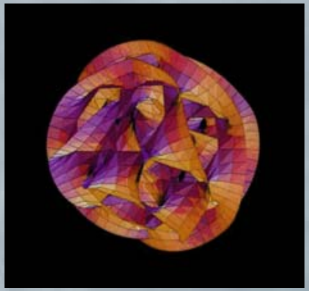
Solitons in Nature

Solitons are one of the most ubiquitous phenomena in physics

- On the tabletop
  - Superconductors, Superfluids, BECs, Quantum Hall Fluids
- In the sky
  - Magnetic Monopoles, Cosmic strings
- In quantum field theory
  - Instantons, Monopoles, Vortices,
  - Duality and Strongly Coupled Phenomena
- In mathematics
  - Integrability of PDEs
  - Topological Invariants







Solitons in Nature

And among the most important….

Instantons

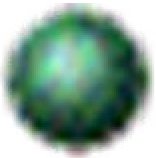
BPST

U(N) Gauge Field

$$S = \int d^4x \; \frac{1}{2e^2} \, \mathrm{Tr} \, F_{\mu\nu} F^{\mu\nu}$$

Instanton Equations: $F_{\mu\nu} = {}^\star F_{\mu\nu}$

Instanton Action: $S = \frac{4\pi^2 k}{e^2}$



- The Instanton is a co-dimension 4 object, localized in (Euclidean) space time

- The scale size of the instanton is a parameter of the solution, a Goldstone mode arising from conformal invariance.

Magnetic Monopoles

't Hooft, Polyakov

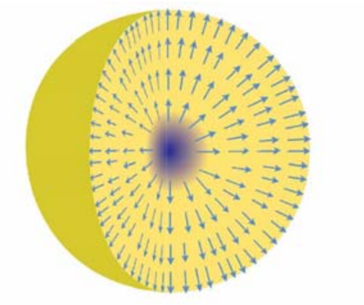
Adjoint Scalar

$$S = \int d^4x \; \frac{1}{2e^2} \, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu \phi)^2$$

Vacuum: $\langle \phi \rangle = \vec{\phi} \cdot \vec{H}$ so that $U(N) \rightarrow U(1)^N$

Monopole Equations: $B_i = \mathcal{D}_i \phi$

Monopole Mass: $M = \frac{4\pi}{e^2} \, \vec{\phi} \cdot \vec{g}$



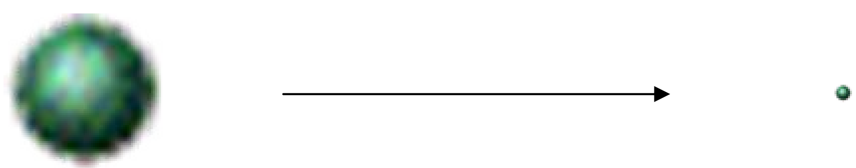
- The monopole is a co-dimension 3, particle-like object

What Happened to the Instanton?

- The conformal invariance of the theory is broken by the vacuum expectation value

$$\langle \phi \rangle = \vec{\phi} \cdot \vec{H}$$

- The instanton shrinks to a pointlike, singular object. No smooth instanton solution now exists.



Another Deformation: The Higgs Phase

Fundamental Scalars

$$S = \int d^4x \; \frac{1}{2e^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2 + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2$$

$$- \sum_{i=1}^{N_f} q_i^\dagger \phi^2 q_i - \frac{e^2}{2}\,\mathrm{Tr}\,\big(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\big)^2$$

D-term, with FI parameter

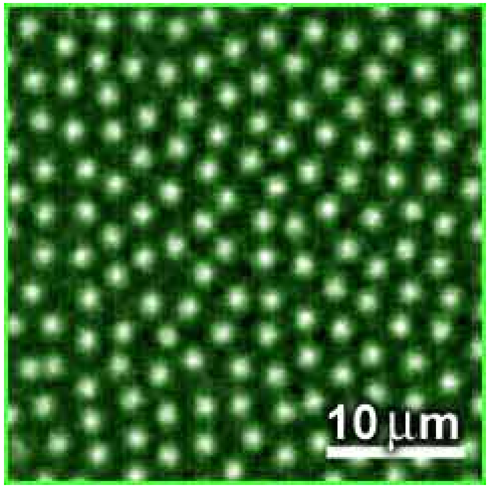
- A zero-energy vacuum requires $N_f \geq N_c$
- We take the vacuum expectation values

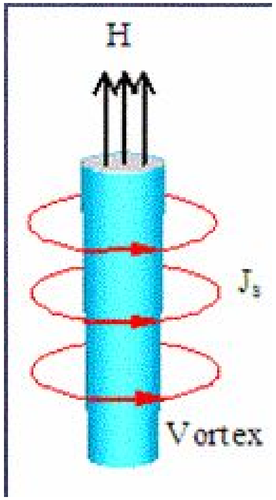
$$\langle\phi\rangle = 0 \qquad\qquad \langle q_i^{\,a}\rangle = v\delta_i^a$$

- The gauge group is now broken completely: $U(N) \to 0$

Vortices in Superconductors

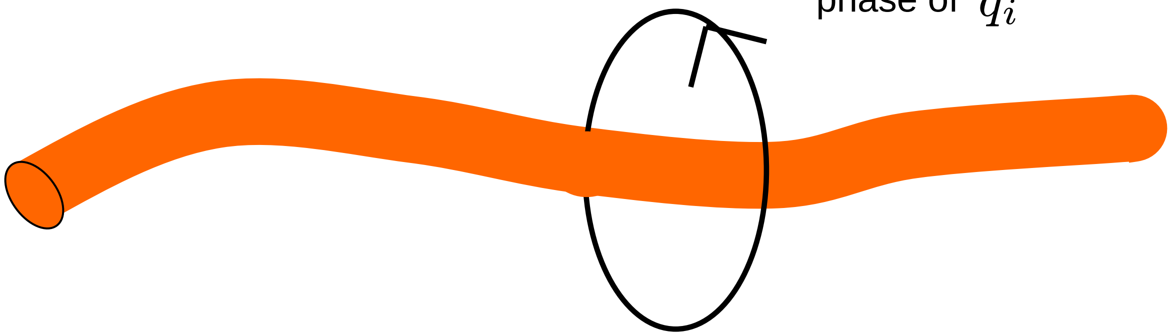
- The theory lies in the Higgs phase. It is like a non-Abelian superconductor.
- In a real superconductor, the Meissner effect means that magnetic flux cannot propagate freely.
- It forms collimated flux tubes, or strings. These are solitons
- They are supported by the winding phase of q.





Vortices

'Nielsen and Olesen,



Vortex Equations: $$(B_3)^a{}_b = e^2(\sum_i q_i^a q_{ib}^\dagger - v^2 \delta^a{}_b)$$

$$(\mathcal{D}_z q_i)^a = 0$$

$z = x_1 + i x_2$

Vortex Tension: $$T_{\text{vortex}} = 2\pi v^2$$

Vortex Moduli Space

Suppose we have an Abelian vortex solution $B_\star$,$q_\star$. We can trivially embed this in the non-Abelian theory. When $N_f = N_c$

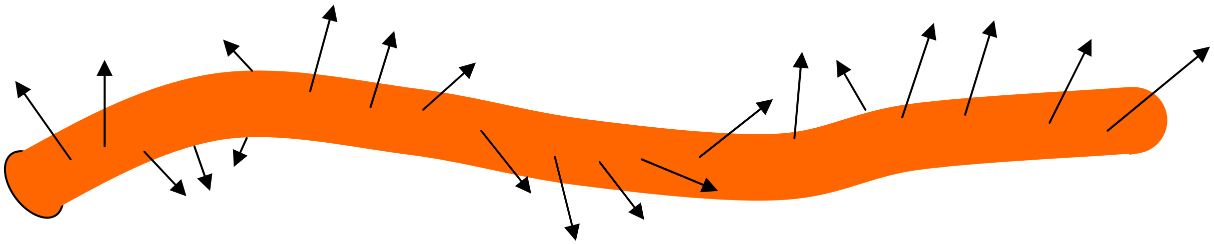
$$B = \begin{pmatrix} B_\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \qquad q = \begin{pmatrix} q_\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}$$

Different embeddings $\Longrightarrow$ moduli space of vortex

$$SU(N)_{\rm diag}/SU(N-1)\times U(1) \cong \mathbf{CP}^{N-1}$$

Vortex Dynamics

The low energy dynamics of an infinite, straight vortex string is the d=1+1 sigma model with target space $\mathbf{C} \times \mathbf{CP}^{N-1}$



Size of $\mathbf{CP}^{N-1}$ is

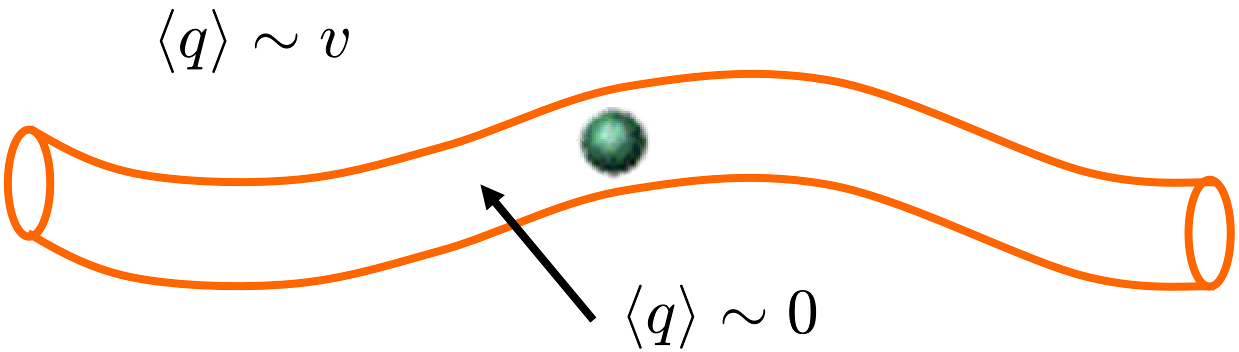
$$r = \frac{2\pi}{e^2}$$

This means that when the 4d theory is weakly coupled, the 2d theory is also weakly coupled.

What Happened to the Instanton Now?

Hanany and Tong, '04

In the Higgs phase, the instanton can nestle inside the vortex string.



The configuration is ¼-BPS

$$F_{12} - F_{34} = \frac{e^2}{2}\left(\sum_i q_i q_i^\dagger - v^2\right)$$

$$F_{14} = F_{23} \qquad F_{13} = F_{24}$$

$$\mathcal{D}_z q_i = 0 \qquad \mathcal{D}_{\bar{w}} q_i = 0$$

The Trapped Instanton

- From the worldvolume perspective of the vortex string, the trapped instanton looks like a sigma-model lump (or worldsheet instanton).
- The sigma-model lump has action

$$S = 2\pi r = \frac{4\pi^2}{e^2} = S_{inst}$$

- The moduli space of lumps is a submanifold of the moduli space of instantons, defined by the fixed point of a U(1) action.

<u>ADHM Construction for Instantons</u>

- N=(4,4) U(k) vector multiplet
- + adjoint hypermultiplet
- + N fund. hypermultiplets

throw away half the fields $\longrightarrow$

<u>AD Construction for Vortices/ Lumps</u>

- N=(2,2) U(k) vector multiplet
- + adjoint chiral multiplet
- + N fund. chiral multiplets
- + N' anti-fund chiral multiplets

Final Deformation: Adding Masses

$$S = \int d^4x \, \frac{1}{2e^2} \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu \phi)^2 + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2$$

$$- \sum_{i=1}^{N_f} q_i^\dagger (\phi - m_i)^2 q_i - \frac{e^2}{2} \mathrm{Tr}\, (\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2)^2$$

add masses

Vacuum: $\langle \phi \rangle = \mathrm{diag}(m_{i_1}, m_{i_2}, \ldots, m_{i_{N_c}})$

and $\langle q \rangle \sim v$

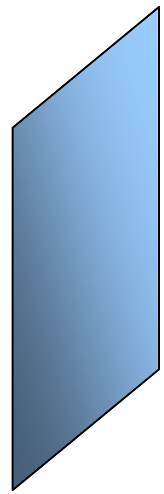
$\Longrightarrow \begin{pmatrix} N_f \\ N_c \end{pmatrix}$ isolated vacua

The theory now has a mass gap for all N_f

Domain Walls

Abraham and Townsend '91

$$\vec{\phi} = \vec{m}$$



$$\vec{\phi} = \vec{m}'$$

Domain Wall Equations: $$\mathcal{D}_3 \phi = \frac{e^2}{2}\left(\sum_i q_i q_i^\dagger - v^2\right)$$

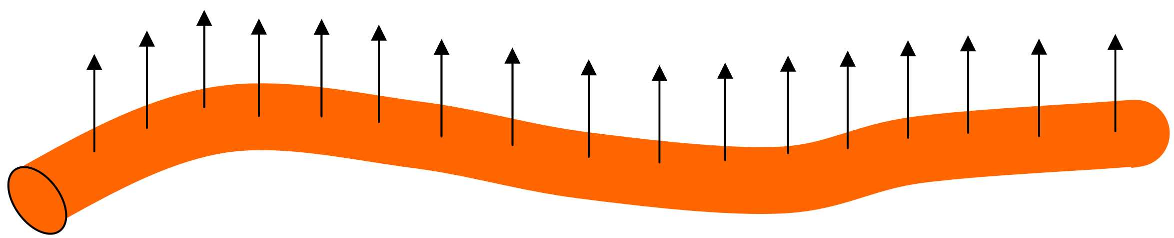
$$\mathcal{D}_3 q_i = (\phi - m_i) q_i$$

Domain Wall Tension: $$T = v^2 \Delta\vec{\phi} \cdot \vec{g}$$

What Became of the Vortices?

- The internal moduli space of vortices is lifted by the masses, leaving behind only isolated, diagonal solutions.

$$B = \begin{pmatrix} 0 & & \\ & B_\star & \\ & & \ddots \end{pmatrix}$$



- From the perspective of the vortex worldsheet, this can be understood as a inducing a potential on $\mathbf{CP}^{N-1}$, leaving N vacua.

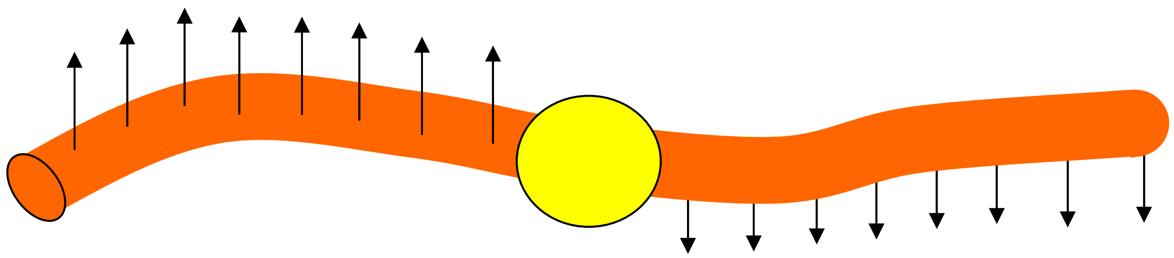
- There is now the possibility of a kink on the worldsheet. It has mass

$$M_{\mathrm{kink}} = r|m_1 - m_2| = \frac{4\pi\,\langle\phi\rangle}{e^2} = M_{\mathrm{mono}}$$

Confined Monopoles

Tong '03

The kink in the vortex worldsheet is a confined magnetic monopole



$$B \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad B \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The configuration is again ¼-BPS:

$$B_1 = \mathcal{D}_1\phi \qquad B_2 = \mathcal{D}_2\phi$$

$$B_3 = \mathcal{D}_3\phi + \frac{e^2}{2}\left(\sum_i q_i q_i^\dagger - v^2\right)$$

$$\mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \qquad \mathcal{D}_3 q_i = -(\phi - m_i)q_i$$

Kinks vs Monopoles

Hanany and Tong, '05

- There is a relationship between the moduli space of domain walls (or kinks) and the moduli space of monopoles
- The moduli space of domain walls is a submanifold of the moduli space of monopoles, defined by the fixed point of a U(1) action.
- The domain wall moduli space contains the information about which walls can pass, and which cannot.

<u>Nahm Construction for Monopoles</u>

$$\mathcal{D}_y X^i - \frac{i}{2}\epsilon^{ijk}[X^j, X^k] = \delta(y)$$

throw away half the fields →

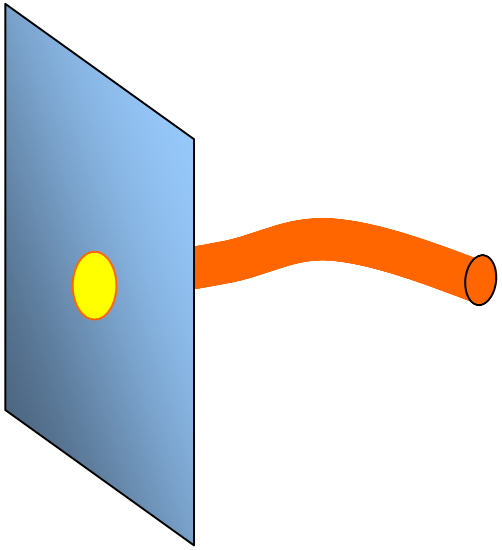
<u>(Almost) Trivial Construction for Kinks</u>

$$\mathcal{D}_y X^i = \delta(y)$$

D-Branes

Gauntlett, Portugues, Tong and Townsend, '00
Shifman and Yung,'03

- The same BPS equations, with different boundary conditions, also have solutions describing vortex strings ending on domain walls

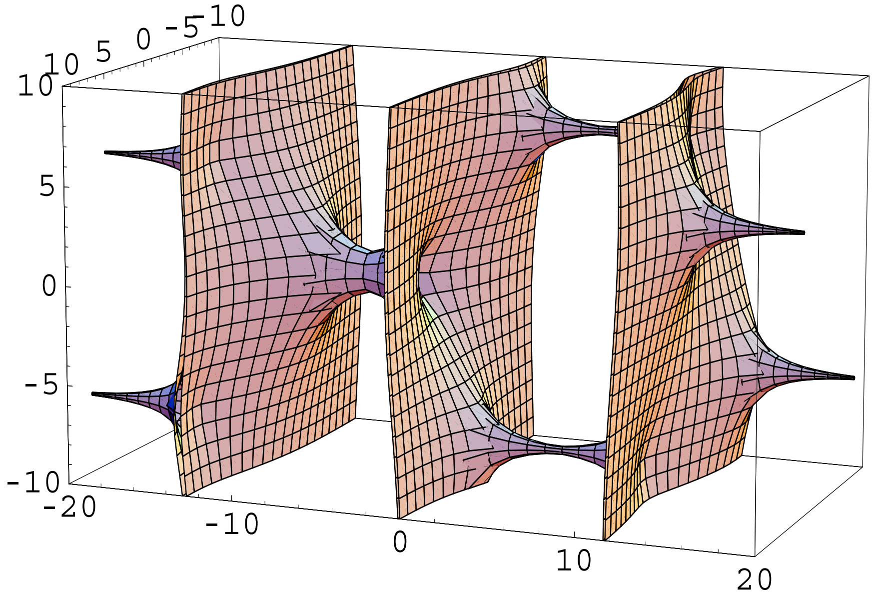


- There is an extra, negative energy density localized where the string meets the wall. It is a **boojum**. It has the negative mass of half a magnetric monopole.

Sakai and Tong '05

D-Branes

- There exists an analytic solution for a single string ending on a domain wall (in the $e^2 \to \infty$ limit). Other numerical solutions have been found.



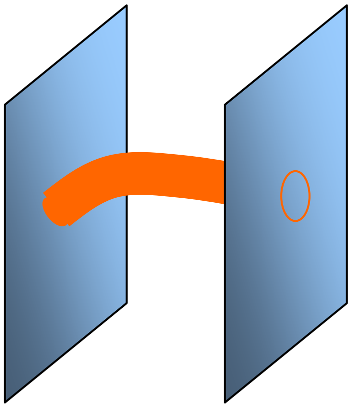
Isozumi, Nitta, Ohashi and Sakai '04

Are They Really D-Branes?

- The most important feature of D-branes is the existence of an open-string description of their dynamics.
- In particular, D-branes in Nature **do not** share this feature.
- For example: The fluid dynamics system describing the weather in Wyoming admits D-brane solitons
- Only a crazy person would suggest an open-string description of cloud dynamics.

D-Branes in Field Theory

Tong '05



The domain walls are D-branes for the vortex string.

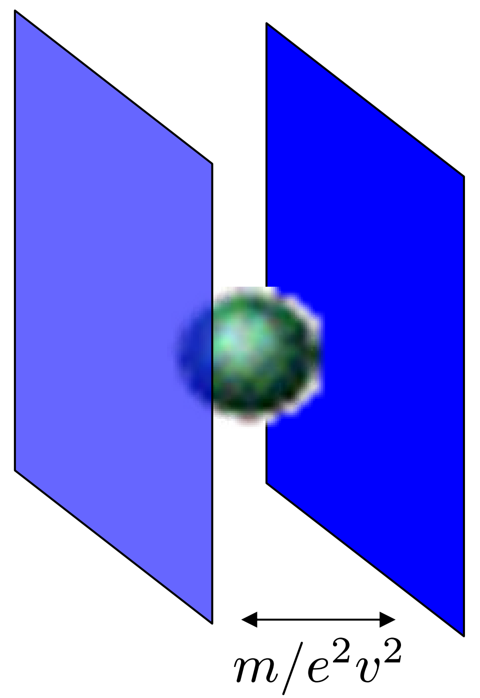
The classical scattering of two domain walls is described by a cigar-like moduli space

There also exists an open string description. The string between two walls gives rise to a chiral multiplet and associated Chern-Simons terms on the walls. The d=2+1 quantum dynamics reproduces the classical scattering of domain walls.

What Happened to the Instantons?

- For generic masses, there is no where for them to hide. They shrink to zero size
- But for degenerate masses, they can nestle inside domain walls

e.g. U(2) with four flavors



This configuration is not BPS

Instantons as Skyrmions

Eto, Nitta, Ohashi, Tong '05

- There is a U(2) flavour symmetry in the field theory which descends to a symmetry on the wall. The collective coordinate of the wall is a U(2) group valued field,

$$g = \mathcal{P} \exp \left(i \int_{-\infty}^{+\infty} dx^3 \ \ A_3(x^3) \right)$$

- The low-energy dynamics of the wall is the Skyrme model, complete with four-derivative term

- Instantons in the bulk become skyrmions on the wall.

- This gives a physical realization of an old idea due to Atiyah and Manton

Summary of Classical BPS Solitons

- Pure Yang-Mills
  - Instanton
- Yang-Mills + Adjoint Scalar
  - Monopole
- Yang-Mills + Massless Fundamental Scalars
  - Vortex String
  - Trapped Instanton
- Yang-Mills + Adjoint Scalar + Massive Fund. Scalars
  - Domain Wall
  - Confined Monopole
  - D-Brane
  - Domain Wall Skyrmion