Non-Relativistic Superconformal Chern-Simons Theories

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Goa 2015

“On Superconformal Anyons”
with Nima Doroud and Carl Turner arXiv: 1511.01491

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erc

A 3d Superconformal Theory

$$S = \int dtd^2x \ \Big\{ i\phi_i^\dagger \mathcal{D}_0\phi_i + i\psi_i^\dagger \mathcal{D}_0\psi_i - \frac{k}{4\pi}\mathrm{Tr}\,\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho)$$
$$-\frac{1}{2m}\Big(\mathcal{D}_a\phi_i^\dagger\,\mathcal{D}_a\phi_i + \mathcal{D}_a\psi_i^\dagger\,\mathcal{D}_a\psi_i - \psi_i^\dagger B\psi_i\Big)$$
$$-\frac{\pi}{mk}\Big((\phi_j^\dagger\phi_i)(\phi_i^\dagger\phi_j) - (\phi_j^\dagger\psi_i)(\psi_i^\dagger\phi_j) + 2(\phi_i^\dagger\phi_j)(\psi_j^\dagger\psi_i)\Big)\Big\}$$

U(N_C) Chern-Simons theory with *N_F* flavours

Leblanc, Lozano and Min ‘93
Nakayama *et. al.* ‘08, ‘09
Lee[3] ‘09

Comments

$$\begin{aligned} S = \int dtd^2x \ \ \Big\{ & i\phi_i^\dagger \mathcal{D}_0\phi_i + i\psi_i^\dagger \mathcal{D}_0\psi_i - \frac{k}{4\pi}\mathrm{Tr}\,\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) \\ & -\frac{1}{2m}\left(\mathcal{D}_a\phi_i^\dagger\,\mathcal{D}_a\phi_i + \mathcal{D}_a\psi_i^\dagger\,\mathcal{D}_a\psi_i - \psi_i^\dagger B\psi_i\right) \\ & -\frac{\pi}{mk}\left((\phi_j^\dagger\phi_i)(\phi_i^\dagger\phi_j) - (\phi_j^\dagger\psi_i)(\psi_i^\dagger\phi_j) + 2(\phi_i^\dagger\phi_j)(\psi_j^\dagger\psi_i)\right)\Big\} \end{aligned}$$

- The particles are anyons
  - “Boson” ϕ has spin -*1/2k*
  - “Fermion” ψ has spin *1/2* - *1/2k*

- The gauge invariant operators need dressing by the dual photon σ

$$\Phi_i = e^{-i\sigma/k}\phi_i \qquad\qquad \Psi_i = e^{-i\sigma/k}\psi_i$$

- The sign of the potential depends on the sign of *k*
  - *k > 0* is a repulsive force between bosons
  - *k < 0* is an attractive force between bosons

Bosonic Symmetries

$$S = \int dt d^2x \ \Big\{ i\phi_i^\dagger \mathcal{D}_0 \phi_i + i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{k}{4\pi} \mathrm{Tr}\, \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho)$$
$$-\frac{1}{2m} \Big(\mathcal{D}_a\phi_i^\dagger \, \mathcal{D}_a\phi_i + \mathcal{D}_a\psi_i^\dagger \, \mathcal{D}_a\psi_i - \psi_i^\dagger B \psi_i \Big)$$
$$-\frac{\pi}{mk} \Big((\phi_j^\dagger \phi_i)(\phi_i^\dagger \phi_j) - (\phi_j^\dagger \psi_i)(\psi_i^\dagger \phi_j) + 2(\phi_i^\dagger \phi_j)(\psi_j^\dagger \psi_i) \Big) \Big\}$$

- Particle numbers: $\mathcal{N}_B = \int d^2x \ \rho_B$ and $\mathcal{N}_F = \int d^2x \ \rho_F$

$$\rho_B = \phi_i^\dagger \phi_i \qquad \rho_F = \psi_i^\dagger \psi_i$$

- Hamiltonian H and momentum $P = \int d^2x \ \mathcal{P}$

- Galilean boosts: $G = \frac{m}{2} \int d^2x \ \bar{z}(\rho_B + \rho_F)$

Bosonic Conformal Symmetries

$$\begin{aligned} S = \int dt d^2x \ \ \Big\{ & i\phi_i^\dagger \mathcal{D}_0 \phi_i + i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{k}{4\pi} \mathrm{Tr}\, \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) \\ & -\frac{1}{2m} \Big(\mathcal{D}_a \phi_i^\dagger \, \mathcal{D}_a \phi_i + \mathcal{D}_a \psi_i^\dagger \, \mathcal{D}_a \psi_i - \psi_i^\dagger B \psi_i \Big) \\ & -\frac{\pi}{mk} \Big((\phi_j^\dagger \phi_i)(\phi_i^\dagger \phi_j) - (\phi_j^\dagger \psi_i)(\psi_i^\dagger \phi_j) + 2(\phi_i^\dagger \phi_j)(\psi_j^\dagger \psi_i) \Big) \Big\} \end{aligned}$$

- Dilatation: $$D = \int d^2x \ (z\mathcal{P} + \bar{z}\bar{\mathcal{P}})$$
- Special Conformal: $$C = \frac{m}{2} \int d^2x \ |z|^2 (\rho_B + \rho_F)$$

SO(2,1) “Schrodinger” algebra

$$[H, C] = -iD \qquad i[D, C] = +2C$$

$$i[D, H] = -2H$$

Primary Operators

We want to compute the dimensions of operators

$$i[D, \mathcal{O}] = -\Delta_{\mathcal{O}} \mathcal{O}$$

From the algebra, it's simple to show that:

- H raises dimension by 2
- P raises the dimension by 1
- G lowers the dimension by 1
- C lowers the dimension by 2

A primary operator sits at the bottom of a tower. It obeys

$$[G_a, \mathcal{O}] = [C, \mathcal{O}] = 0$$

The State-Operator Map

Nishida and Son ’07
(de Alfaro, Fubini and Furlan ‘76)

The Spectrum of *D* on the plane = Spectrum of *H* with Harmonic Trap!

$$L_0 = H + C$$

with $C = \frac{m}{2} \int d^2x \; |z|^2 (\rho_B + \rho_F)$

For primary operator

$$|\Psi_{\mathcal{O}}\rangle = e^{-H} \mathcal{O}(0)|0\rangle$$

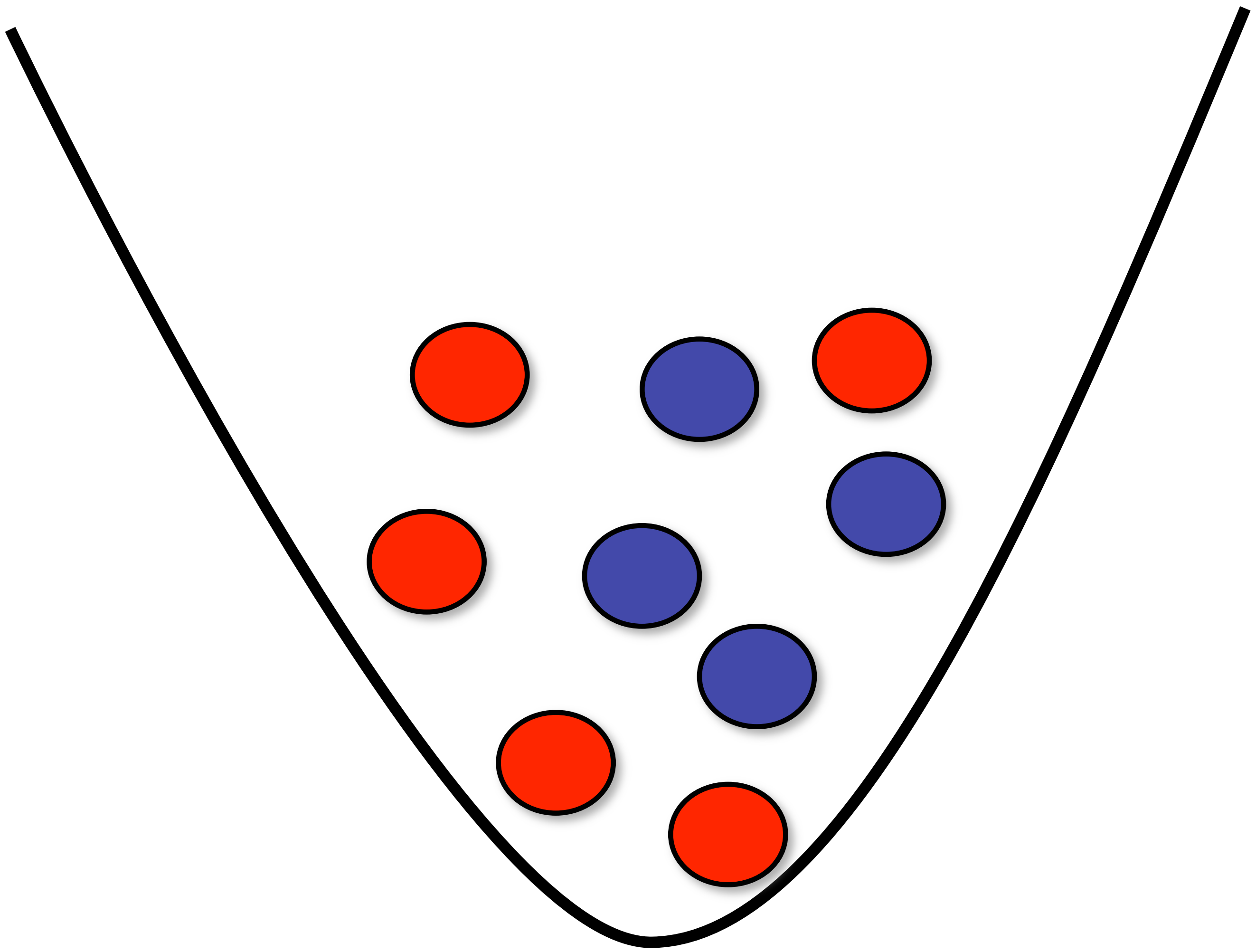
$$i[D, \mathcal{O}] = -\Delta_{\mathcal{O}} \mathcal{O}$$

$$\Longrightarrow$$

$$L_0 |\Psi_{\mathcal{O}}\rangle = \Delta_{\mathcal{O}} |\Psi_{\mathcal{O}}\rangle$$

The Goal

Compute spectrum of primary operators with fixed N_B and N_F



Super(conformal) Symmetries

$$\begin{aligned} S = \int dt d^2x \ \ \Big\{ & i\phi_i^\dagger \mathcal{D}_0 \phi_i + i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{k}{4\pi} \mathrm{Tr}\, \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) \\ & -\frac{1}{2m} \Big(\mathcal{D}_a \phi_i^\dagger \, \mathcal{D}_a \phi_i + \mathcal{D}_a \psi_i^\dagger \, \mathcal{D}_a \psi_i - \psi_i^\dagger B \psi_i \Big) \\ & -\frac{\pi}{mk} \Big((\phi_j^\dagger \phi_i)(\phi_i^\dagger \phi_j) - (\phi_j^\dagger \psi_i)(\psi_i^\dagger \phi_j) + 2(\phi_i^\dagger \phi_j)(\psi_j^\dagger \psi_i) \Big) \Big\} \end{aligned}$$

$$q = i\sqrt{\frac{m}{2}} \int d^2x \ \phi_i^\dagger \psi_i \qquad Q = \sqrt{\frac{2}{m}} \int d^2x \ \phi_i^\dagger \mathcal{D}_{\bar{z}} \psi_i \qquad S = i\sqrt{\frac{m}{2}} \int d^2x \ z\phi_i^\dagger \psi_i$$

susy algebra:

$$\{q, q^\dagger\} = \frac{m}{2}\mathcal{N} \qquad \{Q, Q^\dagger\} = H$$

$$\{S, S^\dagger\} = C$$

Super(conformal) Symmetries

$$\boxed{\begin{aligned} S = \int dt d^2x \ \Big\{ & i\phi_i^\dagger \mathcal{D}_0 \phi_i + i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{k}{4\pi}\mathrm{Tr}\,\epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) \\ & -\frac{1}{2m}\Big(\mathcal{D}_a\phi_i^\dagger\, \mathcal{D}_a\phi_i + \mathcal{D}_a\psi_i^\dagger\, \mathcal{D}_a\psi_i - \psi_i^\dagger B \psi_i \Big) \\ & -\frac{\pi}{mk}\Big((\phi_j^\dagger\phi_i)(\phi_i^\dagger\phi_j) - (\phi_j^\dagger\psi_i)(\psi_i^\dagger\phi_j) + 2(\phi_i^\dagger\phi_j)(\psi_j^\dagger\psi_i) \Big) \Big\} \end{aligned}}$$

$$q = i\sqrt{\frac{m}{2}} \int d^2x \ \phi_i^\dagger \psi_i \qquad Q = \sqrt{\frac{2}{m}} \int d^2x \ \phi_i^\dagger \mathcal{D}_{\bar{z}} \psi_i \qquad S = i\sqrt{\frac{m}{2}} \int d^2x \ z\phi_i^\dagger \psi_i$$

with angular momentum and R-symmetry

$$J = J_0 - \frac{1}{2k}\mathcal{N}_B + \left(\frac{1}{2} - \frac{1}{2k}\right)\mathcal{N}_F$$

$$R = \frac{2k-1}{3k}\mathcal{N}_B - \frac{k+1}{3k}\mathcal{N}_F$$

and…

$$\boxed{\{Q, S^\dagger\} = \frac{i}{2}\left(iD - J + \frac{3}{2}R \right)}$$

(Anti)-Chiral Primary Operators

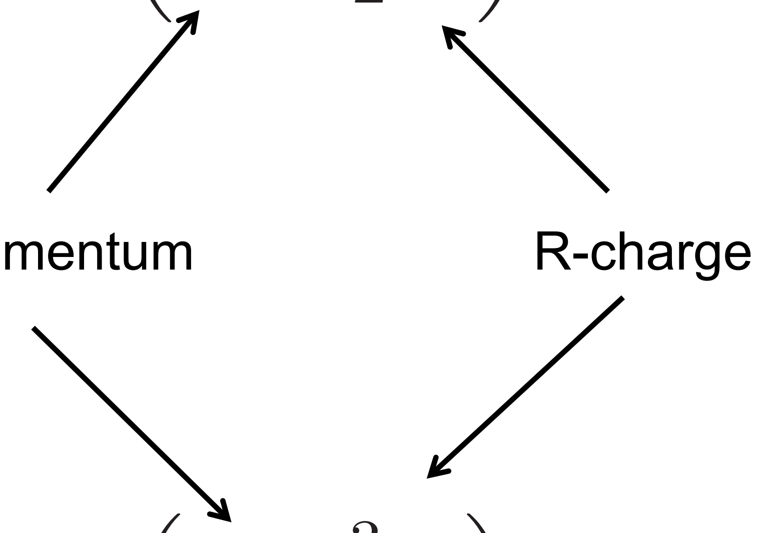
Nakayama ‘08
Lee^3 ‘09

Primary operators sit in supersymmetric multiplets

- Long multiplets have 8 primary operators
- Short multiplets have just 4

A chiral primary operator obeys

$$\Delta_{\mathcal{O}} = -\left(j_{\mathcal{O}} - \frac{3}{2} r_{\mathcal{O}}\right)$$



angular momentum

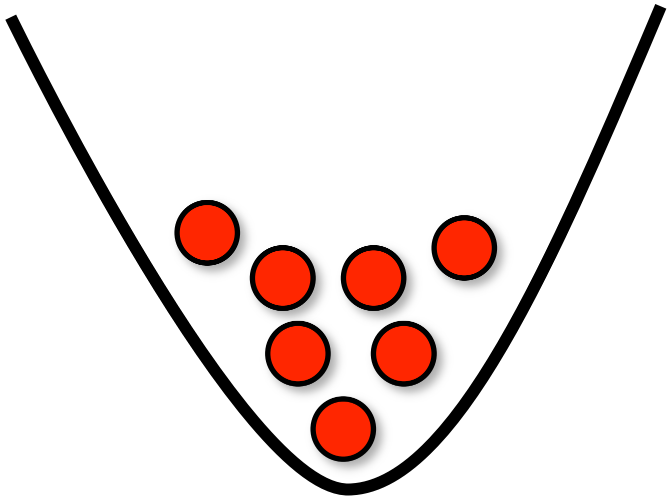
R-charge

An anti-chiral primary operator obeys

$$\Delta_{\mathcal{O}} = +\left(j_{\mathcal{O}} - \frac{3}{2} r_{\mathcal{O}}\right)$$

Chiral Primaries

(for N_C=N_F=1)



The simplest chiral primary creates *n* bosons

$$\mathcal{O} = (\phi^\dagger)^n$$

But there's a surprise with the angular momentum of this operator

$$j_{\mathcal{O}} = -\frac{n^2}{2k}$$

This is a well known result for anyons. (Simplest intuition follows from spin-statistics theorem)

$$\Delta_{\mathcal{O}} = -\left(j_{\mathcal{O}} - \frac{3}{2} r_{\mathcal{O}}\right) \Longrightarrow \boxed{\Delta_{\mathcal{O}} = n + \frac{n(n-1)}{2k}}$$

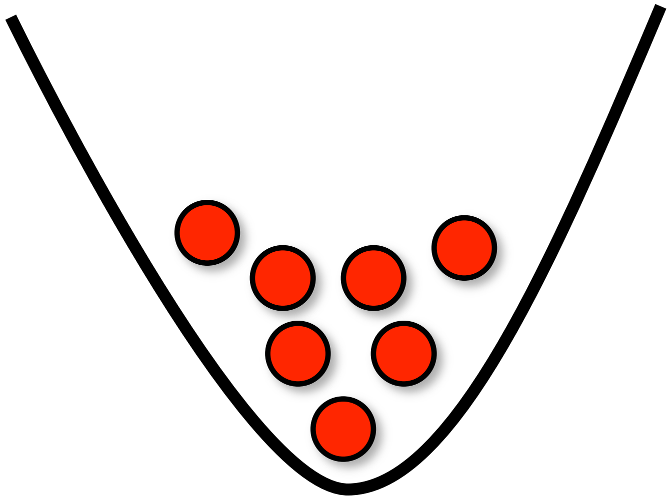
Chiral Primaries

(for N_C=N_F=1)

$\mathcal{O} = (\phi^\dagger)^n$ with $\Delta_{\mathcal{O}} = n + \dfrac{n(n-1)}{2k}$

classical dimension

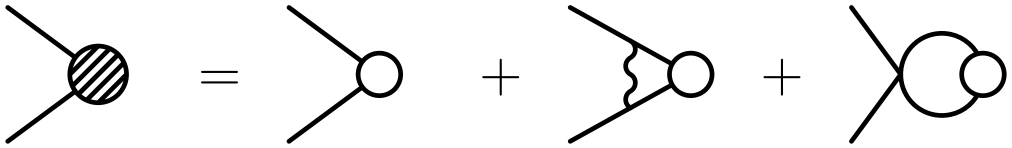
anomalous dimension



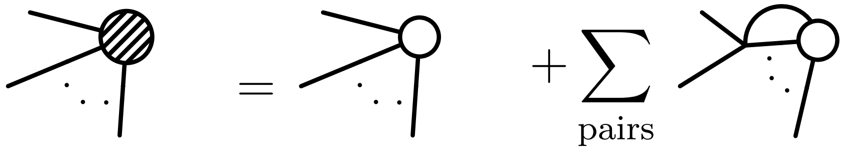
We can compute these one-loop anomalous dimensions explicitly

this diagram has a log divergence

For two particles



and for multi-particles

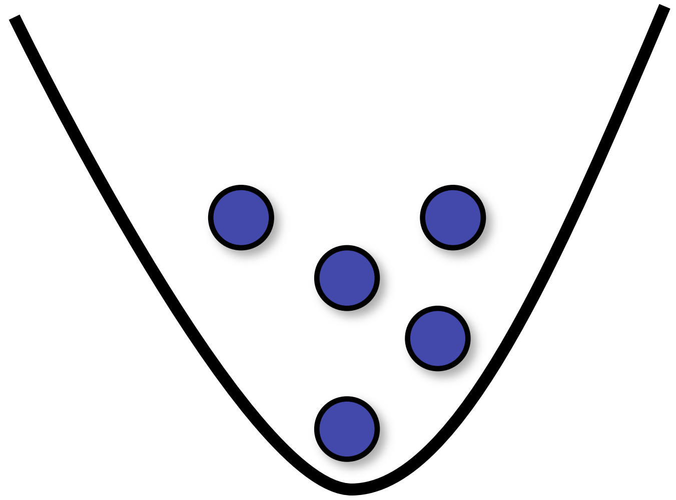


Nishida and Son ‘07

Anti-Chiral Primaries

(for N_C=N_F=*1*)

The simplest fermionic anti-chiral primary is



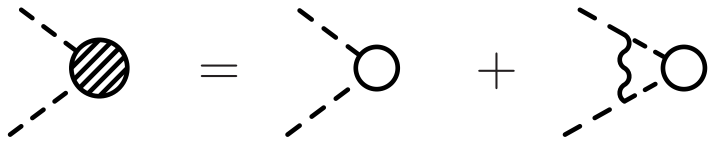
$$\tilde{\mathcal{O}} = \psi^\dagger \partial_{\bar{z}} \psi^\dagger \ldots \partial_{\bar{z}}^{n-1} \psi^\dagger$$

Again, this operator has an unusual angular momentum

$$j_{\tilde{\mathcal{O}}} = \left(\frac{1}{2} - \frac{1}{2k}\right) n^2 = \frac{n}{2} + \frac{n(n-1)}{2} - \frac{n^2}{2k}$$

$$\Delta_{\mathcal{O}} = +\left(j_{\mathcal{O}} - \frac{3}{2} r_{\mathcal{O}}\right) \quad \Longrightarrow \quad \boxed{\Delta_{\tilde{\mathcal{O}}} = \frac{n(n+1)}{2} - \frac{n(n-1)}{2k}}$$

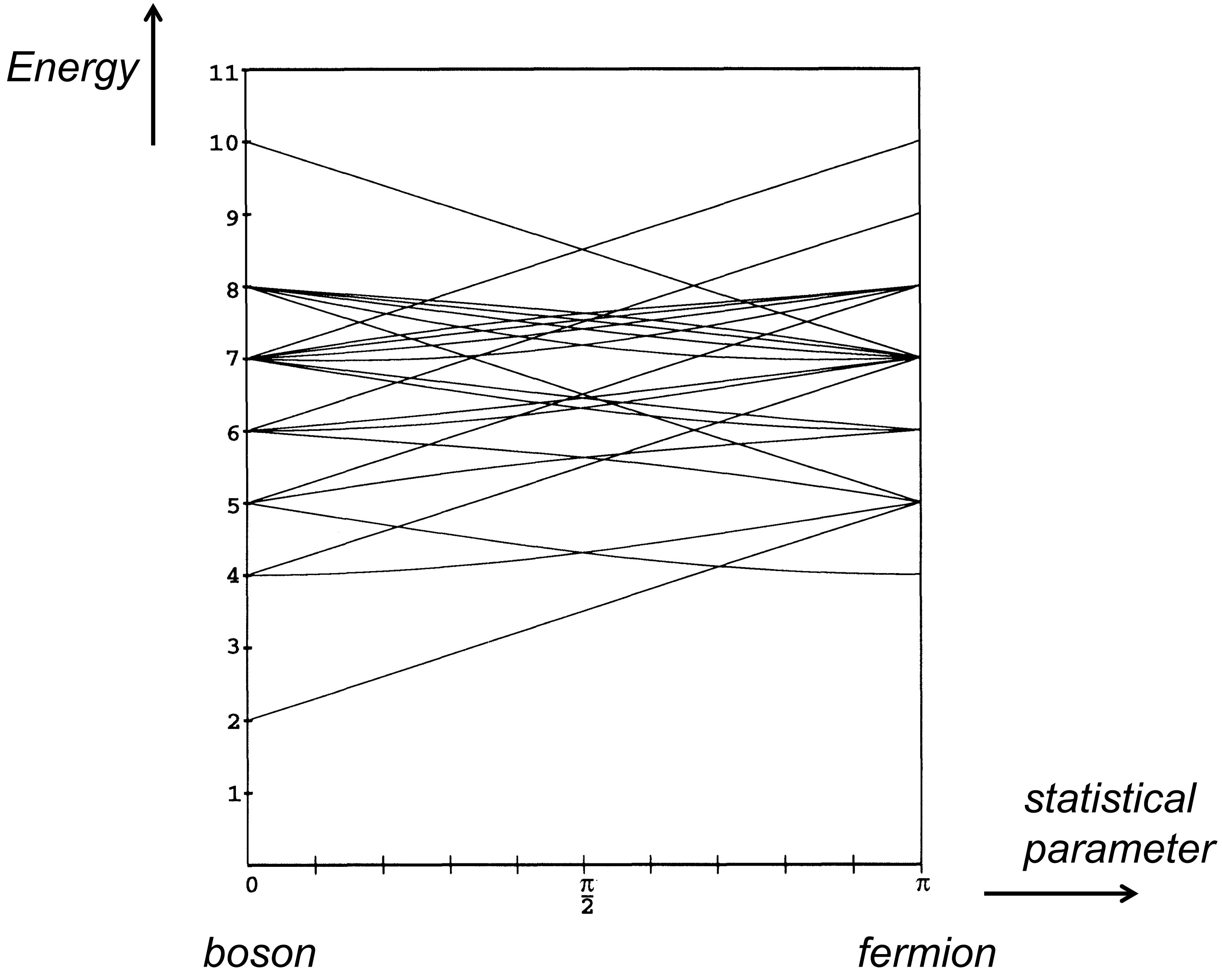
Again, we can compute the anomalous dimension explicitly



A Comparison to Quantum Mechanics

The 3 anyon spectrum has been computed numerically for $k > 0$

Sporre, Verbaarschot and Zahed '91



Similar results are known for the 4 anyon spectrum

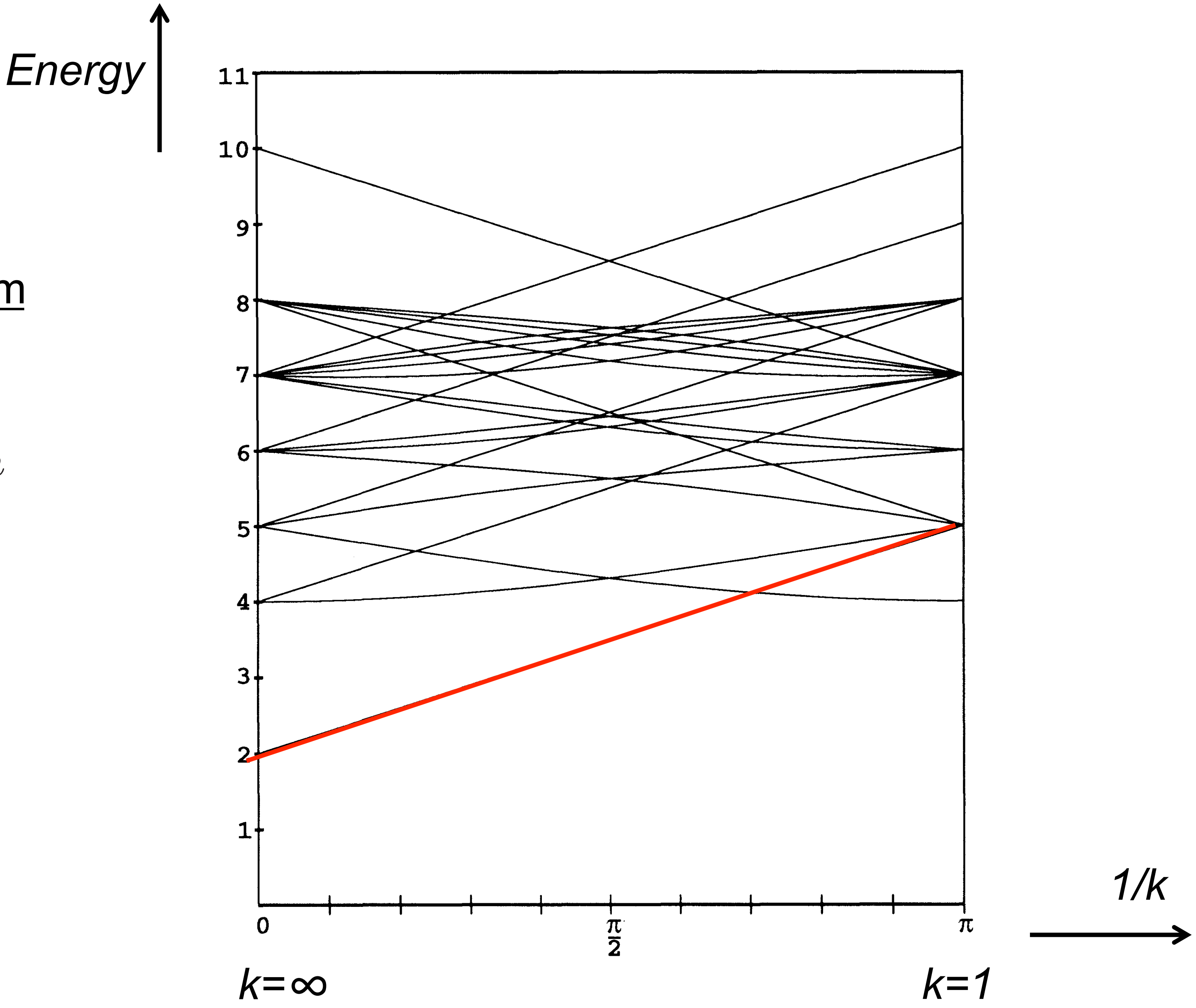
A Comparison to Quantum Mechanics

Sporre, Verbaarschot
and Zahed ‘91

The 3 anyon spectrum has been computed numerically for *k* > *0*

Bosonic spectrum

$$\mathcal{O} = (\phi^\dagger)^n$$



A Comparison to Quantum Mechanics

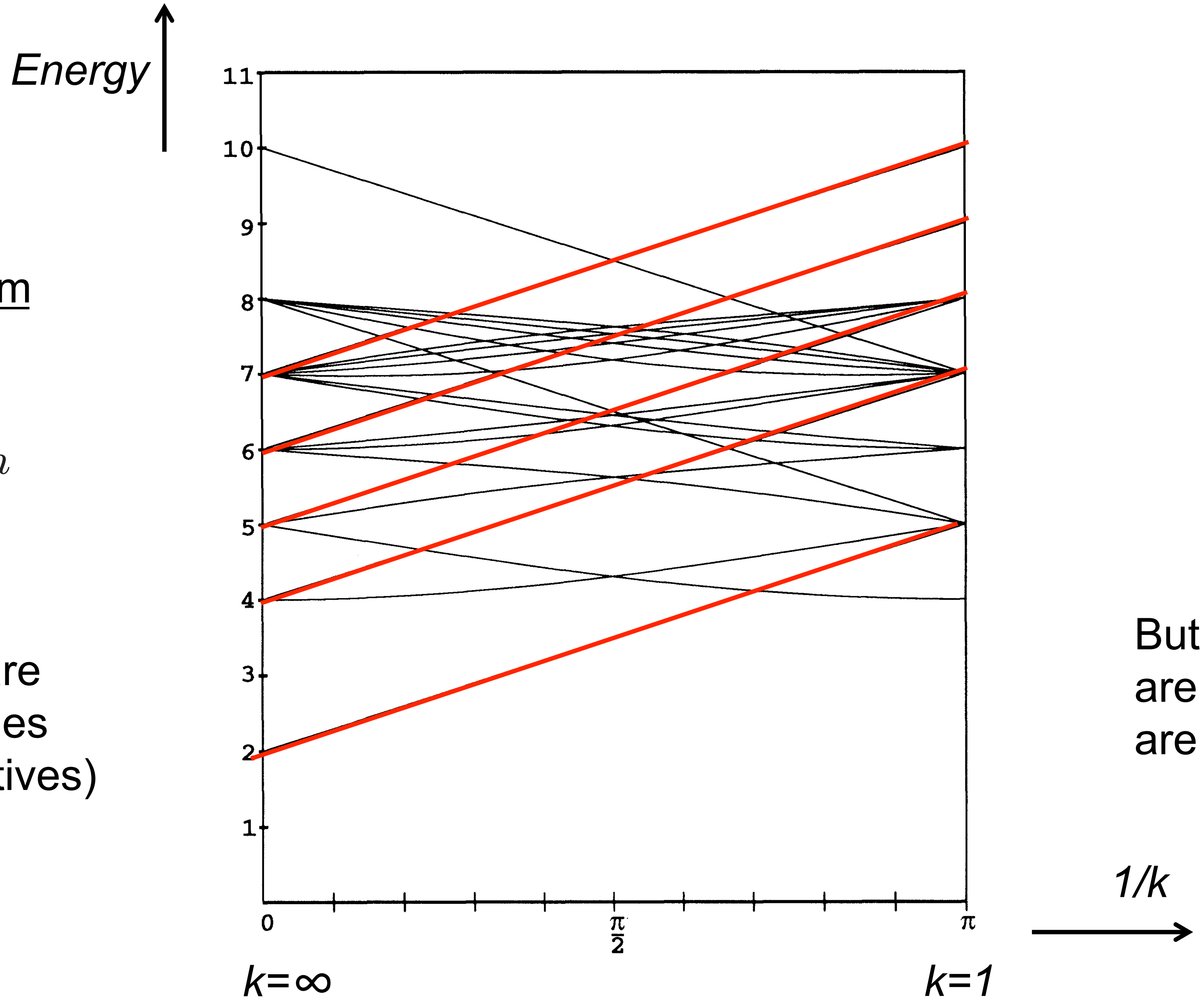
Sporre, Verbaarschot
and Zahed '91

The 3 anyon spectrum has been computed numerically for $k > 0$

<u>Bosonic spectrum</u>

$$\mathcal{O} = (\phi^\dagger)^n$$

All straight lines are
also chiral primaries
(with extra derivatives)



But the interesting states
are those that bend! These
are non-chiral primaries

A Comparison to Quantum Mechanics

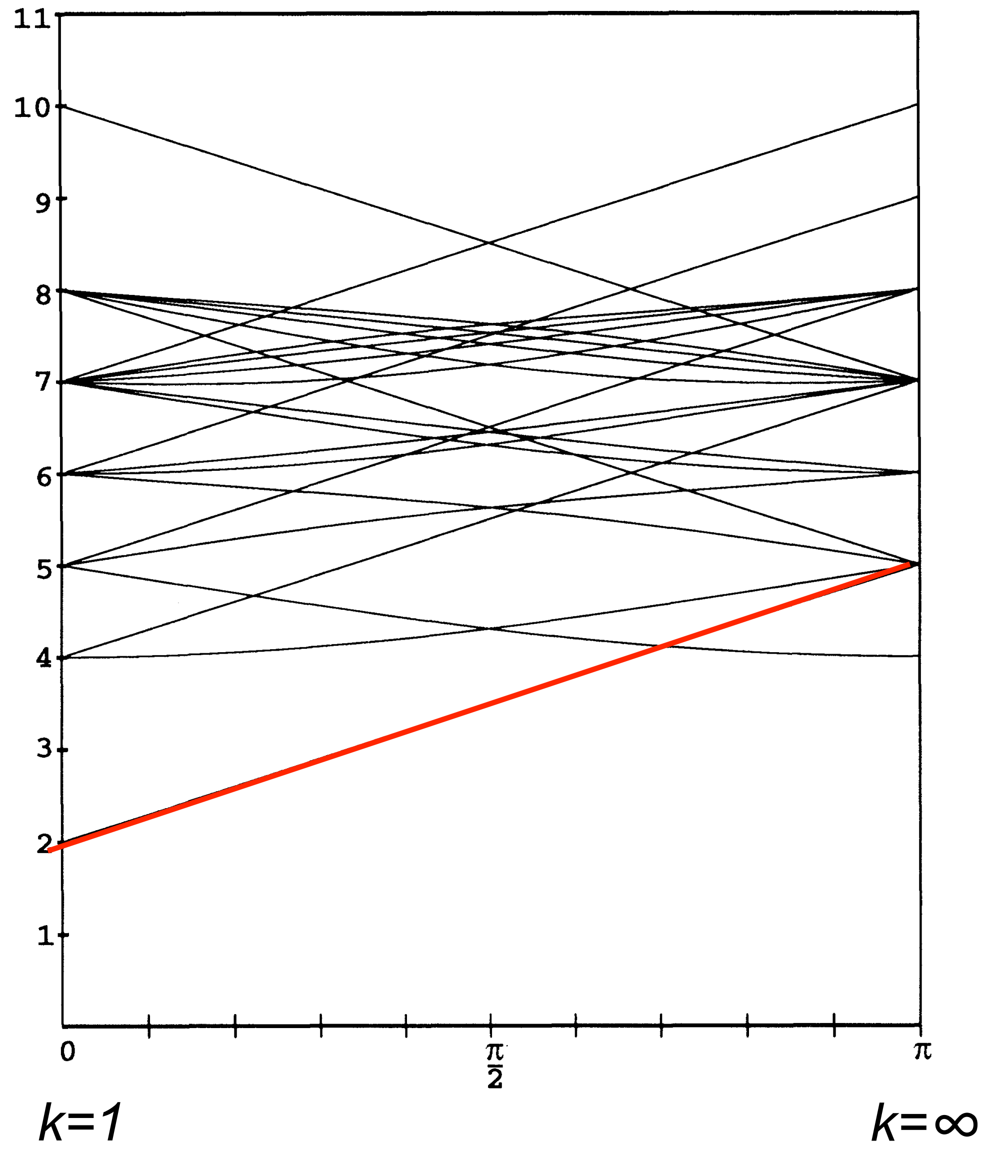
The 3 anyon spectrum has been computed numerically for *k* > *0*

Sporre, Verbaarschot and Zahed '91

<u>Fermionic spectrum</u>

$$\tilde{\mathcal{O}} = \psi^\dagger \partial_{\bar{z}} \psi^\dagger \ldots \partial_{\bar{z}}^{n-1} \psi^\dagger$$

is exactly the same as the bosonic spectrum…



..it just comes at it from the other end

A Unitarity Puzzle

The chiral primaries have

$$\mathcal{O} = (\phi^\dagger)^n \qquad \text{with} \qquad \Delta_{\mathcal{O}} = n + \frac{n(n-1)}{2k}$$

But unitarity requires

$$\Delta_{\mathcal{O}} \geq 1$$

What’s going on when $k < 0$?

We hit the unitarity bound when $n = 2|k|$. We violate it when $n > 2|k|$.

Resolving the Unitarity Puzzle

For fixed number of particles, we can recast the field theory as quantum mechanics

$$H = -\frac{1}{2m}\sum_{i=1}^{n}\left(\partial_a^i + \frac{i}{k}\epsilon_{ab}\,\partial_b^j\,\sum_{j\neq i}\log|\mathbf{x}_i - \mathbf{x}_j|\right)^2 + \frac{2\pi}{mk}\,\sum_{i<j}\delta^2(\mathbf{x}_i - \mathbf{x}_j)$$

Note that *k < 0* $\Longrightarrow$ an attractive delta-function potential between particles

Solve the Schrodinger equation as two particles approach. The ground state has each pair of particles in the S-wave. It is given by

$$\Psi_0 = \prod_{i<j}|\mathbf{x}_i - \mathbf{x}_j|^{-1/|k|} \qquad \Longleftrightarrow \qquad \mathcal{O} = (\phi^\dagger)^n$$

This diverges as two particles approach. The divergence is non-normalisable when *n* ≥ 2|*k*|

Resolving the Unitarity Puzzle

We can match all chiral primary operators to the quantum mechanical wavefuntions

Unitarity violated
$\Delta_{\mathcal{O}} \leq 1$

$\Longleftrightarrow$

Quantum mechanical
wavefunction non-normalisable

A Vortex Puzzle

The theory contains vortices. For $N_C=N_F=1$, the vortices obey the equations

$$B = \frac{2\pi}{k}|\phi|^2 \quad , \quad \mathcal{D}_z\phi = 0$$

These are *Jackiw-Pi* vortices. They are non-topological but BPS. Despite 300 papers on these vortices, no one knows what role they play in the quantum theory. (Including me)

Some tantalising facts:

- Solutions only exist when $k < 0$.
- A single vortex on the plane has particle number $n = 2|k|$.

This is where our operators hit the unitarity bound. Are these vortices new operators/states in the theory? Seems natural, but not at all obvious…

Thank you for your attention