Quantum Vortex Strings

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Plan for the Talk

- Study the quantum dynamics of vortex strings (magnetic flux tubes) in four-dimensional gauge theories
  - The vortex strings contain information about the strongly coupled gauge dynamics in four-dimensions

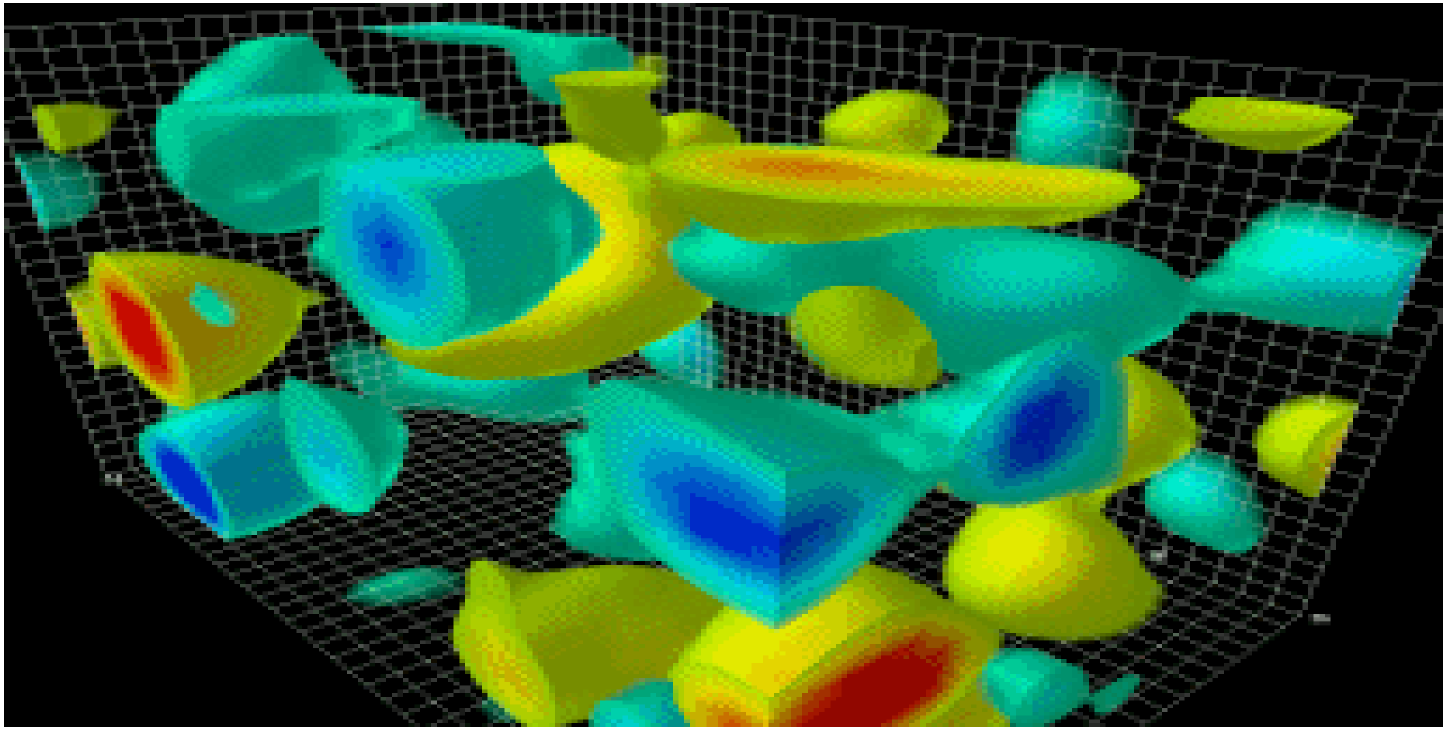
- Based on
  - "Vortices, Instantons and Branes" hep-th/0306150, with Ami Hanany
  - "Monopoles in the Higgs Phase" hep-th/0307302
  - "Vortex Strings and 4d Gauge Dynamics" hep-th/0403158 with Ami Hanany
  - "Superconformal Vortex Strings" hep-th/0610214
  - "Heterotic Vortex Strings" hep-th/0703045 with Mohammad Edalati

The Take-Home Message

- For 25 years we've known that 4d non-abelian gauge theories share certain features with 2d sigma-models
    - Asymptotic freedom
    - Confinement
    - Dynamically generated mass gap
    - Anomalies
    - Instantons
    - Large N limits
- The vortex string provides a *quantitative* link.

A Cartoon of the Basic Idea

- Take a strongly coupled theory with U(N) gauge group and some fundamental scalar fields.



Graphics by Leinweber

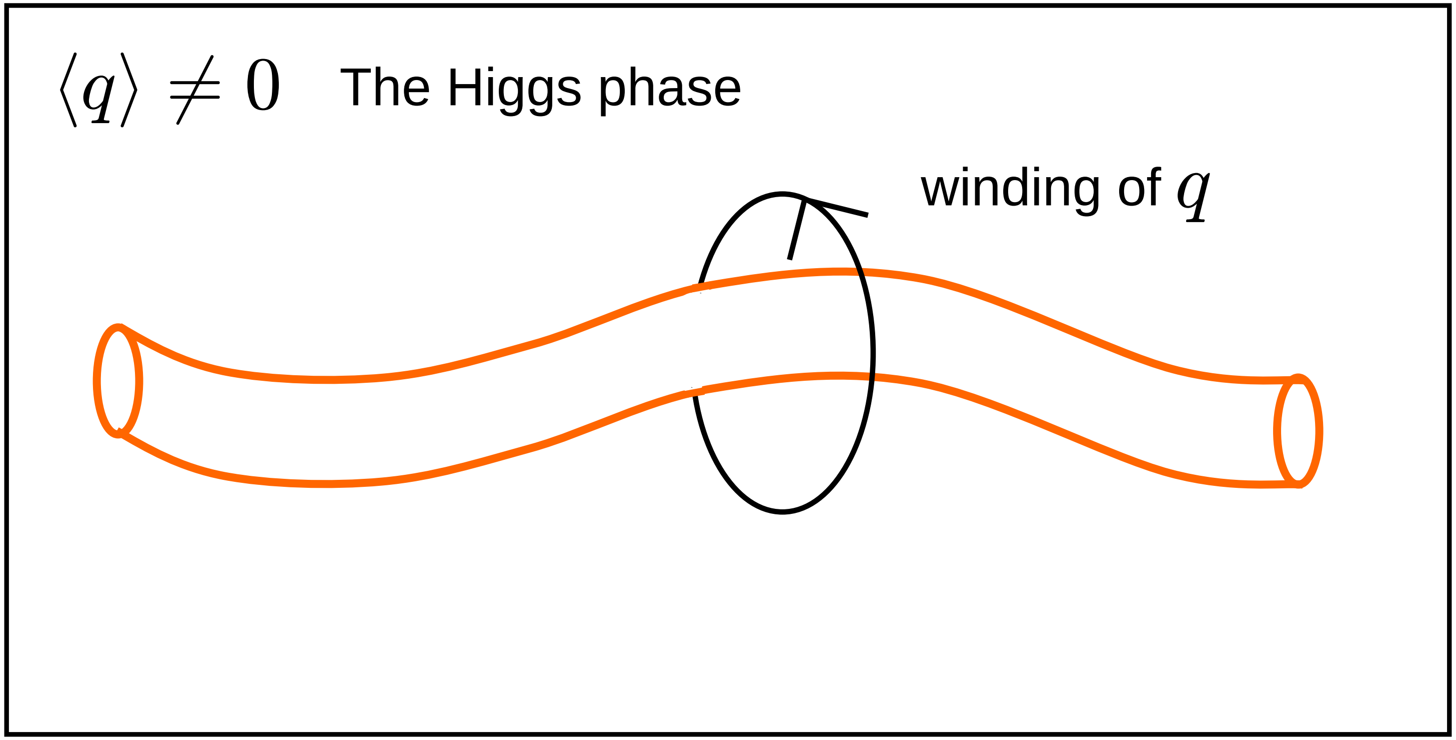
A Cartoon of the Basic Idea

- Deform the theory by inducing an expectation value for the scalar fields
  - If the gauge group is completely broken, the theory now lies in the weakly coupled Higgs phase

$\langle q \rangle \neq 0$ The Higgs phase

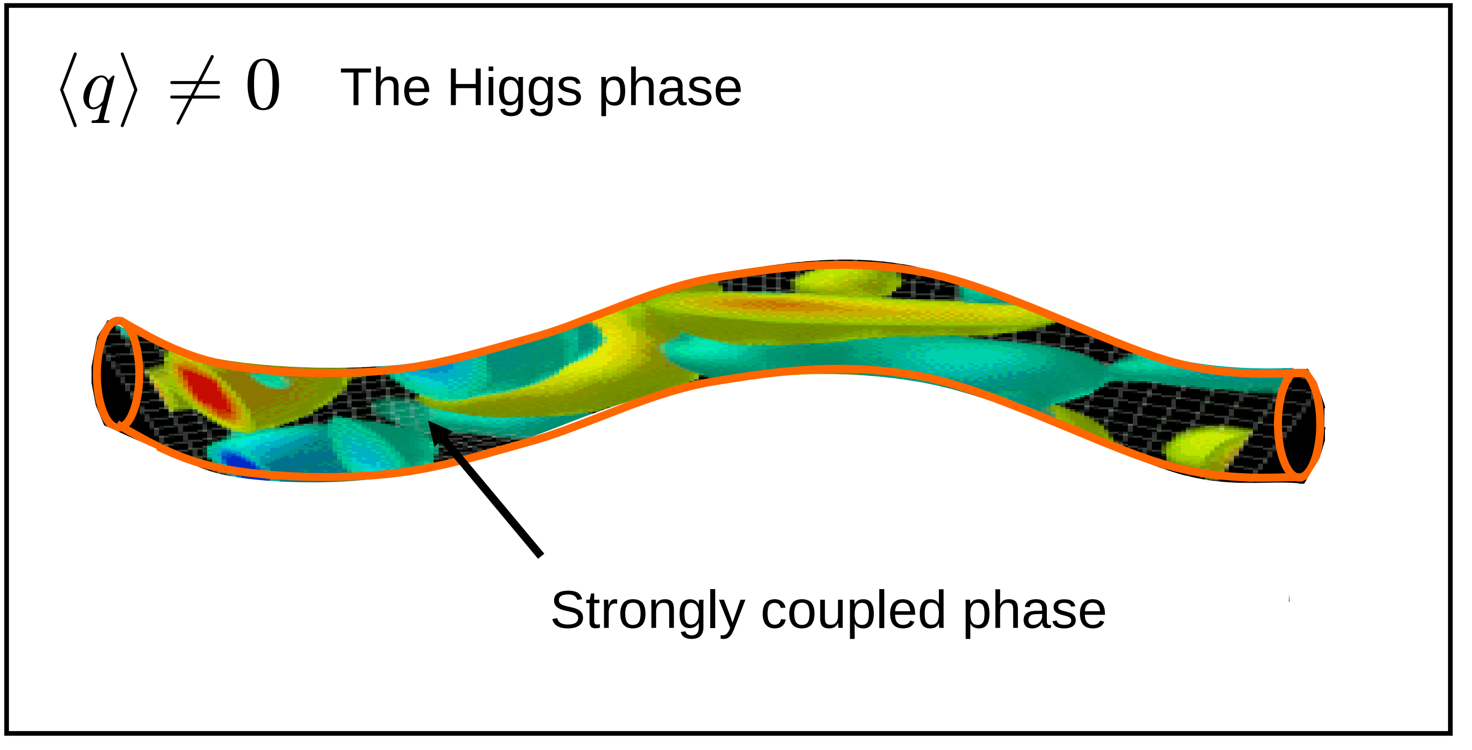
A Cartoon of the Basic Idea

- The theory now admits vortex strings, supported by the phase of the scalar winding at infinity



A Cartoon of the Basic Idea

- The interior of the vortex string is a strongly coupled system
  - The vortex string knows about the original 4d gauge theory.



The 4d Theory

Starting point: d=3+1, $\mathcal{N}$ =2 supersymmetric theory with U(N) gauge group and N_f =N fundamental flavours.

$$L = \frac{1}{4e^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}q_i|^2$$

$$-\frac{e^2}{2}\,\mathrm{Tr}\,\Big(\sum_i q_i \otimes q_i^\dagger - v^2\Big)^2 \ \text{ + fermions + …}$$

We write $q^a{}_i$ where a=1,…,N is the colour index, and i=1,…,N_f is the flavour index.

The 4d Theory

- Vacuum: The ground state is unique (up to a gauge transformation)

$$q^a{}_i = v\,\delta^a{}_i$$

- Spectrum: The theory has a mass gap,with

$$m_\gamma = m_q \sim ev$$

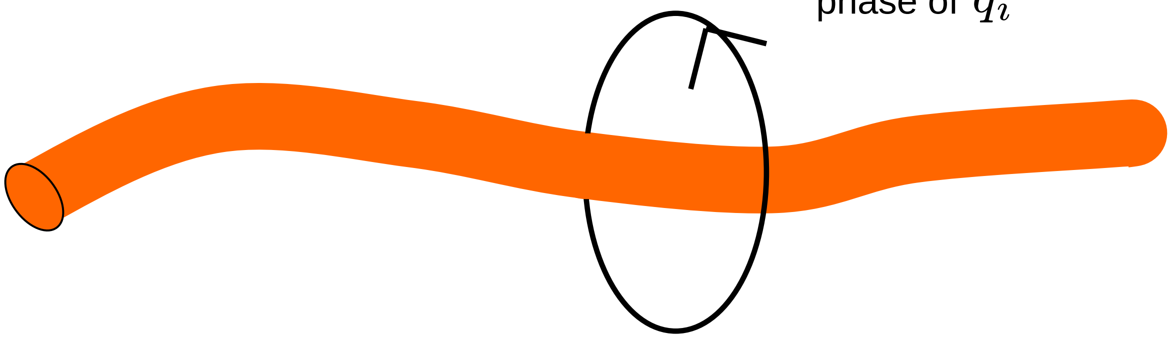
- Symmetries: The theory lies in the “colour-flavour” locked phase

$$U(N)\times SU(N_f)\rightarrow SU(N)_{\rm diag}$$

Note that overall U(1) is broken: $\Pi_1(U(1)) \cong \mathbf{Z}$ $\Longrightarrow$ Vortices

Vortices

Broken U(1) gauge symmetry $\Longrightarrow$ Vortices



$$(B_3)^a{}_b = e^2(\textstyle\sum_i q_i^a q^\dagger_{ib} - v^2 \delta^a{}_b)$$

$$(\mathcal{D}_z q_i)^a = 0$$

$z = x_1 + ix_2$

$$\boxed{T_{\text{vortex}} = 2\pi v^2}$$

Nielsen and Olesen, ‘73

Vortex Moduli Space

Suppose we have an Abelian vortex solution $B_\star, q_\star$. We can trivially embed this in the non-Abelian theory.

$$B = \begin{pmatrix} B_\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \qquad q = \begin{pmatrix} q_\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}$$

Different embeddings $\Longrightarrow$ moduli space of vortex

$$SU(N)_{\rm diag}/SU(N-1)\times U(1) \cong \mathbf{CP}^{N-1}$$

Vortex Dynamics

The low energy dynamics of an infinite, straight vortex string is the d=1+1 sigma model with target space

translational mode $\Rightarrow$ $\boxed{\mathbf{C} \times \mathbf{CP}^{N-1}}$ $\Leftarrow$ internal modes

Size of $\mathbf{CP}^{N-1}$ is $\boxed{r = \frac{2\pi}{e^2}}$

This means that when the 4d theory is weakly coupled, the 2d theory is also weakly coupled.

The fermions in 4d give rise to fermi zero modes on the string. This ensures that the d=1+1 theory is also supersymmetric. It has N=(2,2) susy.

Vortex Dynamics

There is a useful way to write the $\mathbf{CP}^{N-1}$ sigma model in terms of a U(1) gauge theory (often referred to as a "gauged linear sigma model").

$$L_{\text{vortex}} = \frac{1}{2g^2} F_{03}^2 + \sum_{i=1}^{N} |\mathcal{D}\psi_i|^2 - \frac{g^2}{2}\left(\sum_i |\psi_i|^2 - r\right)^2$$

$$+ \text{ fermions}$$

$$\sum_i |\psi_i|^2 = r \mod \psi_i \rightarrow e^{i\alpha}\psi_i \quad \Longrightarrow \quad \mathbf{CP}^{N-1}$$

One advantage of writing things this way is that it's simple to generalize to other situation. For example……

Multiple Vortices

- A solution with flux k has 2kN collective coordinates.

- The dynamics of the vortices is described by an N=(2,2) gauge theory (derived using a D-brane construction)
  - U(k) vector multiplet
  - + adjoint chiral multiplet
  - + N fundamental chiral multiplets.

- The Higgs branch of this theory is identified with the vortex moduli space. It captures
  - The topology
  - The asymptotic metric
  - But the full Kahler metric differs from the moduli space metric.

Vortices vs. Instantons

- Consider k instantons in U(N). The moduli space has dimension 4kN. It is described by the ADHM N=(4,4) gauge theory
  - U(k) vector multiplet
  - adjoint hypermultiplet
  - N fundamental hypermultiplets
- c.f. the N=(2,2) vortex theory
  - U(k) vector multiplet
  - adjoint chiral multiplet
  - N fundamental chiral multiplets
- We learn: the vortex moduli space is a middle dimensional complex submanifold of the instanton moduli space!

Adding Masses

There is a generalization of the vortex theory that will prove useful in exploring how the vortex string captures 4d physics: we add masses for the fundamental fields.

ϕ is a complex adjoint scalar field

$$L = \frac{1}{4e^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2e^2}\,\mathrm{Tr}|\mathcal{D}\phi|^2 + \sum_{i=1}^{N}|\mathcal{D}q_i|^2$$

$$-\sum_i q_i^\dagger(\phi - m_i)^2 q_i - \frac{e^2}{2}\mathrm{Tr}\left(\sum_i q_i \otimes q_i^\dagger - v^2\right)^2$$

+ fermions + …

complex masses

The 4d Theory with Masses

- Vacuum: $q^a{}_i = v\,\delta^a{}_i$ and $\phi = \mathrm{diag}(m_1, \ldots, m_N)$

- Symmetries: $U(N) \times SU(N_f) \xrightarrow{v} SU(N)_{\mathrm{diag}} \xrightarrow{m} U(1)^{N-1}_{\mathrm{diag}}$

- Question: What happens to vortex strings now we've turned on masses?

Vortices with Masses

Old solutions remain solutions only if the new term vanishes:

$$\sum_i q_i^\dagger (\phi - m_i)^2 q_i = 0 \quad \Longrightarrow \quad \text{all solutions diagonal}$$

$$B = \begin{pmatrix} 0 & B_\star & & \\ & \ddots & & \\ & & & 0 \end{pmatrix} \qquad q = \begin{pmatrix} v & q_\star & & \\ & \ddots & & \\ & & & v \end{pmatrix}$$

$\Longrightarrow$ We have N different types of vortex solutions

Note: before we could sweep out a moduli space of vortices using the SU(N) global symmetry. But this is lifted to U(1) factors by the masses.

Vortex Dynamics with Masses

The vortex dynamics is now described by a sigma model with potential.
In the U(1) gauge theory we introduce a new auxiliary field σ

$$L_{\mathrm{vortex}} = \frac{1}{2g^2} F_{03}^2 + \frac{1}{2g^2} \partial\sigma^2 + \sum_{i=1}^{N} |\mathcal{D}\psi_i|^2$$
$$- \sum_i (\sigma - m_i)^2 |\psi_i|^2 - \frac{g^2}{2} \left(\sum_i |\psi_i|^2 - r \right)^2$$

+ fermions

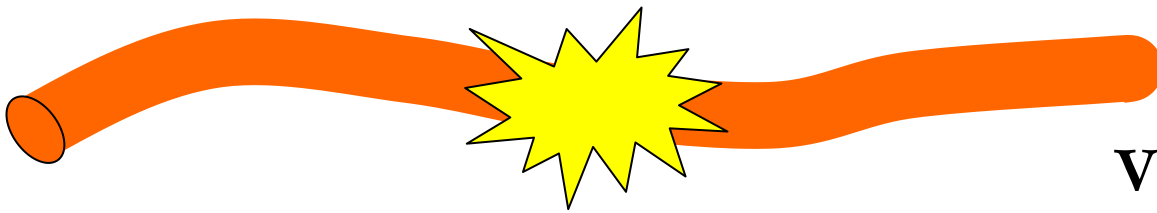
The vortex theory has N vacua $\sigma = m_i$, $|\psi_j|^2 = r\delta_{ij}$

$\Longleftrightarrow$ N different types of vortex flux tube

Kinky Vortex Strings

Isolated vacua in the vortex worldsheet $\Rightarrow$ kinks in the string

Vacuum 1:



Vacuum 2:

$$B \sim \begin{pmatrix} \mathbf{1} & \\ & \mathbf{0} \end{pmatrix} \qquad\qquad B \sim \begin{pmatrix} \mathbf{0} & \\ & \mathbf{1} \end{pmatrix}$$

The kink is a source of magnetic flux $B \sim \begin{pmatrix} \mathbf{-1} & \\ & \mathbf{1} \end{pmatrix}$. This is the flux carried by a 't Hooft-Polyakov monopole. Moreover, the mass of the kink is given by

$$M_{\rm kink} = r|m_1 - m_2| = \frac{4\pi\,\langle\phi\rangle}{e^2} = M_{\rm mono}$$

$$\boxed{\text{The kink is a magnetic monopole, confined by the Meissner effect}}$$

The Confined Monopole

- The kink on the string is a confined 't Hooft-Polyakov monopole
- The full d=3+1 configuration is not known, but can be shown to obey the following *first* order differential equations. They are ¼-BPS.

$$B_1 = \mathcal{D}_1\phi \qquad B_2 = \mathcal{D}_2\phi \qquad \mathcal{D}_1 q_i = i\mathcal{D}_2 q_i$$

$$B_3 = \mathcal{D}_3\phi + e^2\left(\textstyle\sum_i q_i q_i^\dagger - v^2\right) \qquad \mathcal{D}_3 q_i = -(\phi - m_i) q_i$$

- There's more: a vortex inside the vortex string corresponds to a Yang-Mills instanton trapped inside the vortex.
  - This gives a physical explanation for the relationship between the instanton and vortex moduli spaces we saw earlier

The Quantum Theory

Dorey,
Hanany and Tong
Shifman and Yung

The first hint that the vortex string knows about the 4d quantum theory comes from the beta functions. The relationship

$$\boxed{r = \frac{2\pi}{e^2}}$$

is preserved under RG flow of the 2d and 4d theories

$$r(\mu) = r_0 - \frac{N_c}{2\pi}\log\left(\frac{M_{UV}}{\mu}\right)$$

The strong coupling scale Λ of the string worldsheet is the same as that of the unbroken 4d theory.

The Quantum Spectrum

The result $\boxed{M_{\rm kink} = M_{\rm mono}}$ also holds in the full quantum theory.

In the regime $m \gg \Lambda$ both the 2d theory and the unbroken 4d theory are weakly coupled. The mass the solitons has an expansion

$$M = M_{\rm classical} + M_{1-{\rm loop}} + \sum_{n=1}^{\infty} M_{\rm n-instanton}$$

You can compute these quantum corrections in 2d or in 4d, summing over 2d instantons or 4d instantons. They agree!

Reason: The vortex in the vortex string is a trapped Yang-Mills instanton

More on the Quantum Theory

- The elementary string excitations correspond to W-boson and quark fields. The masses in 4d and on the worldsheet coincide. 4d dyons correspond to excited “Q-kinks”

- The 2d twisted superpotential is

$$\tilde{W}(\Sigma) = -\frac{1}{2\pi}\sum_i(\Sigma - m_i)\left[\log\left(\frac{\Sigma - m_i}{\mu}\right) - 1\right] - (r + i\theta)\Sigma$$

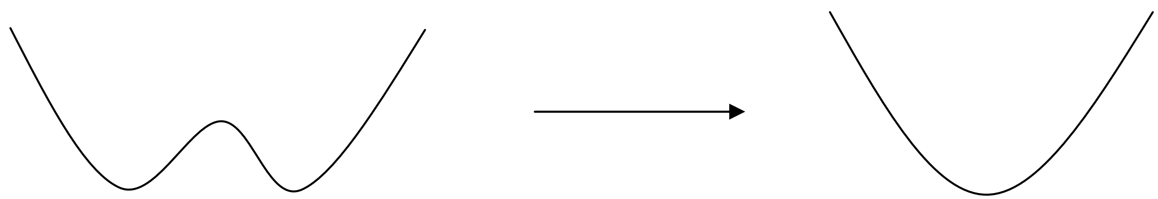
- This coincides with the Seiberg-Witten curve of the 4d theory, which is given by

$$y^2 = \left(\frac{\partial \tilde{W}(x)}{\partial x}\right)^2$$

Superconformal Points

- The 4d theory has special points on its moduli space where both magnetic and electric charges become massless.
- These are 4d conformal field theories known as "Argyres-Douglas" points.

- Question: What happens to the vortex string at these points? The kink on the worldsheet becomes massless



$$V \sim (x-a)^2(x+a)^2 \qquad\qquad V \sim x^4$$

- The worldsheet theory of the string also becomes conformal

Superconformal Strings

- The effective twisted superpotential on the string is

$$\tilde{W} = -\frac{1}{2\pi}\sum_i (\Sigma - m_i)\left[\log\left(\frac{\Sigma - m_i}{\mu}\right) - 1\right] - (r + i\theta)\Sigma$$

- Tune the classical potential generated by masses to cancel the quantum potential

$$m_k = -\exp\left(\frac{2\pi i k}{N}\right)\,\Lambda$$

- The effective superpotential becomes

$$\tilde{W} = c\,\frac{\Sigma^{N+1}}{\Lambda^N} + \ldots$$

- This is the $\mathcal{N} = (2,2)$ A_{N-1} minimal model
- Dimensions of chiral primary operators in 4d and 2d SCFTs coincide

Summary

- Vortex Strings provide a map between four-dimensional gauge theories and two-dimensional sigma models

N=2 theories in 4d		N=(2,2) theories in 2d
Monopole Dyon W-boson, quark	Exact Agreement of BPS mass spectrum	Kink Dyonic Kink Elementary Excitation
Argyres-Douglas Points	Superconformal Field Theories	Minimal Models