TASI Lectures on Solitons

Lecture 4: Domain Walls

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Abstract: Our final lecture is on domain walls (also known as kinks). We do all the usual stuff: solutions, moduli spaces, brane constructions. We focus in the applications on the relationship between kinks and monopoles, leading to a quantitative correspondence between 2d sigma models and 4d gauge theories. Further applications to the 2d black hole, and 3d Chern-Simons-Higgs theories are also described.

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4. Domain Walls

So far we've considered co-dimension 4 instantons, co-dimension 3 monopoles and co-dimension 2 vortices. We now come to co-dimension 1 domain walls, or kinks as they're also known. While BPS domain walls exist in many supersymmetric theories (for example, in Wess-Zumino models [1]), there exists a special class of domain walls that live in gauge theories with 8 supercharges. They were first studied by Abraham and Townsend [2] and have rather special properties. These will be the focus of this lecture. As we shall explain below, the features of these domain walls are inherited from the other solitons we've met, most notably the monopoles.

4.1 The Basics

To find domain walls, we need to deform our theory one last time. We add masses m_i for the fundamental scalars q_i. Our Lagrangian is that of a $U(N_c)$ gauge theory, coupled to a real adjoint scalar field ϕ and N_f fundamental scalars q_i

$$S = \int d^4x \, \mathrm{Tr}\left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2\right) + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2$$
$$- \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,1_N)^2 \qquad (4.1)$$

Notice the way the masses mix with ϕ, so that the true mass of each scalar is $|\phi - m_i|$. Adding masses in this way is consistent with $\mathcal{N} = 2$ supersymmetry. We'll pick all masses to be distinct and, without loss of generality, choose

$$m_i < m_{i+1} \qquad (4.2)$$

As in Lecture 3, there are vacua with $V = 0$ only if $N_f \geq N_c$. The novelty here is that, for $N_f > N_c$, we have multiple isolated vacua. Each vacuum is determined by a choice of N_c distinct elements from a set of N_f

$$\Xi = \{\xi(a) : \ \xi(a) \neq \xi(b) \text{ for } a \neq b\} \qquad (4.3)$$

where $a = 1, \ldots, N_c$ runs over the color index, and $\xi(a) \in \{1, \ldots, N_f\}$. Let's set $\xi(a) < \xi(a+1)$. Then, up to a Weyl transformation, we can set the first term in the potential to vanish by

$$\phi = \mathrm{diag}(m_{\xi(1)}, \ldots, m_{\xi(N_c)}) \qquad (4.4)$$

This allows us to turn on the particular components $q^a{}_i \sim \delta^a{}_{i=\xi(a)}$ without increasing the energy. To cancel the second term in the potential, we require

$$q^a{}_i = v\delta^a{}_{i=\xi(a)} \tag{4.5}$$

The number of vacua of this type is

$$N_{\rm vac} = \begin{pmatrix} N_f \\ N_c \end{pmatrix} = \frac{N_f!}{N_c!(N_f - N_c)!} \tag{4.6}$$

Each vacuum has a mass gap in which there are N_c^2 gauge bosons with $M_\gamma^2 = e^2v^2 + |m_{\xi(a)} - m_{\xi(b)}|^2$, and $N_c(N_f - N_c)$ quark fields with mass $M_q^2 = |m_{\xi(a)} - m_i|^2$ with $i \notin \Xi$.

Turning on the masses has explicitly broken the $SU(N_f)$ flavor symmetry to

$$SU(N_f) \rightarrow U(1)_F^{N_f-1} \tag{4.7}$$

while the $U(N_c)$ gauge group is also broken completely in the vacuum. (Strictly speaking it is a combination of the $U(N_c)$ gauge group and $U(1)_F^{N_f-1}$ that survives in the vacuum).

4.2 Domain Wall Equations

The existence of isolated vacua implies the existence of a domain wall, a configuration that interpolates from a given vacuum Ξ_- at $x^3 \rightarrow -\infty$ to a distinct vacuum Ξ_+ at $x^3 \rightarrow +\infty$. As in each previous lecture, we can derive the first order equations satisfied by the domain wall using the Bogomoln'yi trick. We'll chose x^3 to be the direction transverse to the wall, and set $\partial_0 = \partial_1 = \partial_3 = 0$ as well as $A_0 = A_1 = A_2 = 0$. The tension of the domain wall can be written as [39]

$$\begin{aligned} T_{\rm wall} = & \int dx^3\ \frac{1}{e^2}{\rm Tr}\left(\mathcal{D}_3\phi + \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2)\right)^2 - \mathcal{D}_3\phi\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2) \\ & + \sum_{i=1}^{N_f}\Big(|\mathcal{D}_3 q_i + (\phi - m_i)q_i|^2 - q_i^\dagger(\phi - m_i)\mathcal{D}_3 q_i - \mathcal{D}_3 q_i^\dagger(\phi - m_i)q_i\Big) \\ \geq & \ v^2\ [{\rm Tr}\phi]^{+\infty}_{-\infty} \end{aligned} \tag{4.8}$$

With our vacua Ξ_- and Ξ_+ at left and right infinity, we have the tension of the domain wall bounded by

$$T_{\rm wall} \geq v^2[{\rm Tr}\phi]^{+\infty}_{-\infty} = v^2 \sum_{i\in\Xi_+} m_i - v^2 \sum_{i\in\Xi_-} m_i \tag{4.9}$$

and the minus signs have been chosen so that this quantity is positive (if this isn't the case we must swap left and right infinity and consider the anti-wall). The bound is saturated when the domain wall equations are satisfied,

$$\mathcal{D}_3\phi = -\frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2) \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{4.10}$$

Just as the monopole equations $\mathcal{D}\phi = B$ arise as the dimensional reduction of the instanton equations $F = {}^\star F$, so the domain wall equations (4.10) arise from the dimensional reduction of the vortex equations. To see this, we look for solutions to the vortex equations with $\partial_2 = 0$ and relabel $x^1 \to x^3$ and $(A_1, A_2) \to (A_3, \phi)$. Finally, the analogue of turning on the vev in going from the instanton to the monopole, is to turn on the masses m_i in going from the vortex to the domain wall. These can be thought of as a "vev" for $SU(N_f)$ the flavor symmetry.

4.2.1 An Example

The simplest theory admitting a domain wall is $U(1)$ with $N_f = 2$ scalars q_i. The domain wall equations are

$$\partial_3\phi = -\frac{e^2}{2}(|q_1|^2 + |q_2|^2 - v^2) \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{4.11}$$

We'll chose $m_2 = -m_1 = m$. The general solution to these equations is not known. The profile of the wall depends on the value of the dimensionless constant $\gamma = e^2v^2/m^2$. For $\gamma \ll 1$, the wall can be shown to have a three layer structure, in which the q_i fields decrease to zero in the outer layers, while ϕ interpolates between its two expectation values at a more leisurely pace [3]. The result is a domain wall with width $L_{\rm wall} \sim m/e^2v^2$. Outside of the wall, the fields asymptote exponentially to their vacuum values.

In the opposite limit $\gamma \gg 1$, the inner segment collapses and the two outer layers coalesce, leaving us with a domain wall of width $L_{\rm wall} \sim 1/m$. In fact, if we take the limit $e^2 \to \infty$, the first equation (4.11) becomes algebraic while the second is trivially solved. We find the profile of the domain wall to be [4]

$$q_1 = \frac{v}{A}\, e^{-m(x_3 - X) + i\theta} \quad , \quad q_2 = \frac{v}{A}\, e^{+m(x_3 - X) - i\theta} \tag{4.12}$$

where $A^2 = e^{-2m(x_3 - X)} + e^{+2m(x_3 - X)}$.

The solution (4.12) that we've found in the $e^2 \to \infty$ limit has two collective coordinates, X and θ. The former is simply the position of the domain wall in the transverse

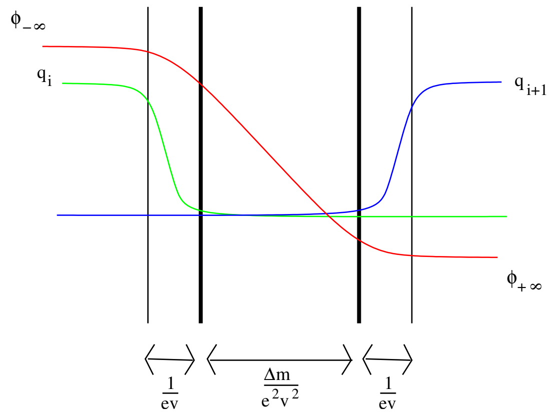

Figure 1: The three layer structure of the domain wall when $e^2v^2 \ll m^2$.

x^3 direction. The latter is also easy to see: it arises from acting on the domain wall with the $U(1)_F$ flavor symmetry of the theory [2]:

$$U(1)_F : q_1 \to e^{i\theta} q_1 \quad , \quad q_2 \to e^{-i\theta} q_2 \tag{4.13}$$

In each vacuum, this coincides with the $U(1)$ gauge symmetry. However, in the interior of the domain wall, it acts non-trivially, giving rise to a phase collective coordinate θ for the solution. It can be shown that X and θ remain the only two collective coordinates of the domain wall when we return to finite e^2 [5].

4.2.2 Classification of Domain Walls

So we see above that the simplest domain wall has two collective coordinates. What about the most general domain wall, characterized by the choice of vauca Ξ_- and Ξ_+ at left and right infinity. At first sight it appears a little daunting to classify these objects. After all, a strict classification of the topological charge requires a statement of the vacuum at left and right infinity, and the number of vacua increases exponentially with N_f. To ameliorate this sense of confusion, it will help to introduce a coarser classification of domain walls which will capture some information about the topological sector, without specifying the vacua completely. This classification, introduced in [6], will prove most useful when relating our domain walls to the other solitons we've met previously. To this end, define the N_f-vector

$$\vec{m} = (m_1, \ldots, m_{N_f}) \tag{4.14}$$

We can then write the tension of the domain wall as

$$T_{\text{wall}} = v^2 \vec{g} \cdot \vec{m} \tag{4.15}$$

which defines a vector $\vec{g}$ that contains entries 0 and ± 1 only. Following the classification of monopoles in Lecture 2, let's decompose this vector as

$$\vec{g} = \sum_{i=1}^{N_f} n_i \vec{\alpha}_i \tag{4.16}$$

with $n_i \in \mathbf{Z}$ and the $\vec{\alpha}_i$ the simple roots of $su(N_f)$,

$$\begin{aligned} \vec{\alpha}_1 &= (1, -1, 0, \ldots, 0) \\ \vec{\alpha}_2 &= (0, 1, -1, \ldots, 0) \\ \vec{\alpha}_{N_f-1} &= (0, \ldots, 0, 1, -1) \end{aligned}$$

Since the vector $\vec{g}$ can only contain 0's, 1's and -1's, the integers n_i cannot be arbitrary. It's not hard to see that this restriction means that neighboring n_i's are either equal or differ by one: $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$.

4.3 The Moduli Space

A choice of $\vec{g}$ does not determine a choice of vacua at left and right infinity. Nevertheless, domain wall configurations which share the same $\vec{g}$ share certain characteristics, including the number of collective coordinates. The collective coordinates carried by a given domain wall was calculated in a number of situations in [7, 8, 9]. Using our classification, the index theorem tells us that there are solutions to the domain wall equations (4.10) only if $n_i \geq 0$ for all i. Then the number of collective coordinates is given by [6],

$$\dim \mathcal{W}_{\vec{g}} = 2 \sum_{i=1}^{N_f-1} n_i \tag{4.17}$$

where $\mathcal{W}_{\vec{g}}$ denotes the moduli space of any set of domain walls with charge $\vec{g}$. Again, this should be looking familiar! Recall the result for monopoles with charge $\vec{g}$ was $\dim(\mathcal{M}_{\vec{g}}) = 4 \sum_a n_a$. The interpretation of the result (4.17) is, as for monopoles, that there are $N_f - 1$ elementary types of domain walls associated to the simple roots $\vec{g} = \vec{\alpha}_i$. A domain wall sector in sector $\vec{g}$ then splits up into $\sum_i n_i$ elementary domain walls, each with its own position and phase collective coordinate.

4.3.1 The Moduli Space Metric

The low-energy dynamics of multiple, parallel, domain walls is described, in the usual fashion, by a sigma-model from the domain wall worldvolume to the target space is

$\mathcal{W}_{\vec{g}}$. As with other solitons, the domain walls moduli space $\mathcal{W}_{\vec{g}}$ inherits a metric from the zero modes of the solution. In notation such that $q = q^a{}_i$ is an $N_c \times N_f$ matrix, the linearized domain wall equations (4.10)

$$\begin{aligned} \mathcal{D}_3\delta\phi - i[\delta A_3, \phi] &= -\frac{e^2}{2}(\delta q\, q^\dagger + q\delta q^\dagger) \\ \mathcal{D}_3\delta q - i\delta A_3\, q &= -(\phi\delta q + \delta\phi\, q - \delta q\, m) \end{aligned} \tag{4.18}$$

where $m = \mathrm{diag}(m_1, \ldots, m_{N_f})$ is an $N_f \times N_f$ matrix. Again, these are to be supplemented by a background gauge fixing condition,

$$\mathcal{D}_3\delta A_3 - i[\phi, \delta\phi] = i\frac{e^2}{2}(q\delta q^\dagger - \delta q\, q^\dagger) \tag{4.19}$$

and the metric on the moduli space $\mathcal{W}_{\vec{g}}$ is defined by the overlap of these zero modes,

$$g_{\alpha\beta} = \int dx^3 \;\mathrm{Tr}\left(\frac{1}{e^2}\left[\delta_\alpha A_3\, \delta_\beta A_3 + \delta_\alpha\phi\, \delta_\beta\phi\right] + \delta_\alpha q\, \delta_\beta q^\dagger + \delta_\beta q\, \delta_\alpha q^\dagger\right) \tag{4.20}$$

By this stage, the properties of the metric on the soliton moduli space should be familiar. They include.

- The metric is Kähler.
- The metric is smooth. There is no singularity as two domain walls approach each other.
- The metric inherits a $U(1)^{N-1}$ isometry from the action of the unbroken flavor symmetry (4.7) acting on the domain wall.

4.3.2 Examples of Domain Wall Moduli Spaces

Let's give some simple examples of domain wall moduli spaces.

One Domain Wall

We've seen that a single elementary domain wall $\vec{g} = \vec{\alpha}_1$ (for example, the domain wall described above in the theory with $N_c = 1$ and $N_f = 2$) has two collective coordinates: its center of mass X and a phase θ. The moduli space is

$$\mathcal{W}_\alpha \cong \mathbf{R} \times \mathbf{S}^1 \tag{4.21}$$

The metric on this space is simple to calculate. It is

$$ds^2 = (v^2\vec{m}\cdot\vec{g})\; dX^2 + v^2\, d\theta^2 \tag{4.22}$$

with the phase collective coordinate living in $\theta \in [0, 2\pi)$.

Two Domain Walls

We can't have two domain walls of the same type, say $\vec{g} = 2\vec{\alpha}_1$, since there is no choice of vacua that leads to this charge. Two elementary domain walls must necessarily be of different types, $\vec{g} = \vec{\alpha}_i + \vec{\alpha}_j$ for $i \neq j$. Let's consider $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$.

The moduli space is simplest to describe if the two domain walls have the same mass, so $\vec{m} \cdot \vec{\alpha}_a = \vec{m} \cdot \vec{\alpha}_b$. The moduli space is

$$\mathcal{W}_{\vec{\alpha}_1+\vec{\alpha}_2} \cong \mathbf{R} \times \frac{\mathbf{S}^1 \times \mathcal{M}_{\rm cigar}}{\mathbf{Z}_2} \tag{4.23}$$

where the interpretation of the $\mathbf{R}$ factor and $\mathbf{S}^1$ factor are the same as before. The relative moduli space has the topology and asymptotic form of a cigar, The relative

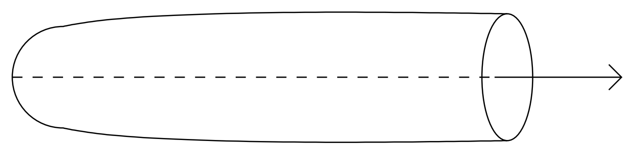

Figure 2: The relative moduli space of two domain walls is a cigar.

separation between domain walls is denoted by R. The tip of the cigar, $R = 0$, corresponds to the two domain walls sitting on top of each other. At this point the relative phase of the two domain walls degenerates, resulting in a smooth manifold. The metric on this space has been computed in the $e^2 \to \infty$ limit, although it's not particularly illuminating [4] and gives a good approximation to the metric at large finite e^2 [10]. Asymptotically, it deviates from the flat metric on the cylinder by exponentially suppressed corrections e^{-R}, as one might expect since the profile of the domain walls is exponentially localized.

4.4 Dyonic Domain Walls

You will have noticed that, rather like monopoles, the domain wall moduli space includes a phase collective coordinate $\mathbf{S}^1$ for each domain wall. For the monopole, excitations along this $\mathbf{S}^1$ give rise to dyons, objects with both magnetic and electric charges. For domain walls, excitations along this $\mathbf{S}^1$ also give rise to dyonic objects, now carrying both topological (kink) charge and flavor charge. Abraham and Townsend called these objects "Q-kinks" [2].

First order equations of motion for these dyonic domain walls may be obtained by completing the square in the Lagrangian (4.1), now looking for configurations that depend on both x^0 and x^3, allowing for a non-zero electric field F_{03}. We have

$$\begin{aligned} T_{\rm wall} = \int dx^3 \ \frac{1}{e^2}{\rm Tr}\left(\cos\alpha\,\mathcal{D}_3\phi + \frac{e^2}{2}(\sum_{i=1}^{N_f}|q_iq_i^\dagger - v^2)\right)^2 - \cos\alpha\,\mathcal{D}_3\phi\,(\sum_{i=1}^{N_f}q_iq_i^\dagger - v^2) \\ +\sum_{i=1}^{N_f}\Big(|\mathcal{D}_3q_i + \cos\alpha\,(\phi - m_i)q_i|^2 - \cos\alpha\,(q_i^\dagger(\phi-m_i)\mathcal{D}_3q_i + \text{ h.c.})\Big) \\ +\frac{1}{e^2}{\rm Tr}(F_{03} - \sin\alpha\,\mathcal{D}_3\phi)^2 + \frac{1}{e^2}\sin\alpha\,F_{03}\mathcal{D}_3\phi \\ +\sum_{i=1}^{N_f}\Big(|\mathcal{D}_0q_i + i\sin\alpha(\phi - m_i)q_i|^2 - \sin\alpha(iq_i^\dagger(\phi-m_i)\mathcal{D}_0q_i + \text{ h.c})\Big) \end{aligned}$$

As usual, insisting upon the vanishing of the total squares yields the Bogomoln'yi equations. These are now to augmented with Gauss' law,

$$\mathcal{D}_3F_{03} = ie^2\sum_{i=1}^{N_f}(q_i\mathcal{D}_0q_i^\dagger - (\mathcal{D}_0q_i)q_i^\dagger) \tag{4.24}$$

Using this, we may re-write the cross terms in the energy-density to find the Bogomoln'yi bound,

$$T_{\rm wall} \geq \pm v^2[{\rm Tr}\phi]^{+\infty}_{-\infty}\,\cos\alpha + (\vec{m}\cdot\vec{S})\sin\alpha \tag{4.25}$$

where $\vec{S}$ is the Noether charge associated to the surviving $U(1)^{N_f-1}$ flavor symmetry, an N_f-vector with $i^{\rm th}$ component given by

$$S_i = i(q_i\mathcal{D}_0q_i^\dagger - (\mathcal{D}_0q_i)q_i^\dagger) \tag{4.26}$$

Maximizing with respect to α results in the Bogomoln'yi bound for dyonic domain walls,

$$\mathcal{H} \geq \sqrt{v^4(\vec{m}\cdot\vec{g})^2 + (\vec{m}\cdot\vec{S})^2} \tag{4.27}$$

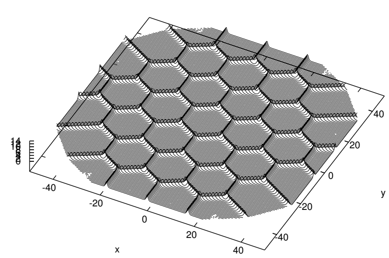

Figure 3:

This square-root form is familiar from the spectrum of dyonic monopoles that we saw in Lecture 2. More on this soon. For now, some further comments, highlighting the some similarities between dyonic domain walls and monopoles.

- There is an analog of the Witten effect. In two dimensions, where the domain walls are particle-like objects, one may add a theta term of the form θF_{01}. This induces a flavor charge on the domain wall, proportional to its topological charge, $\vec{S} \sim \vec{g}$ [11].

- One can construct dyonic domain walls with $\vec{g}$ and $\vec{S}$ not parallel if we turn on complex masses and, correspondingly, consider a complex adjoint scalar ϕ [12, 13]. The resulting 1/4 and 1/8-BPS states are the analogs of the 1/4-BPS monopoles we briefly mentioned in Lecture 2.

- The theory with complex masses also admits interesting domain wall junction configurations [14, 15]. Most notably, Eto et al. have recently found beautiful webs of domain walls, reminiscent of (p, q)5-brane webs of IIB string theory, with complicated moduli as the strands of the web shift, causing cycles to collapse and grow [16, 17]. Examples include the intricate honeycomb structure shown in figure 3 (taken from [16]).

Other aspects of these domain walls were discussed in [18, 19, 20, 21].

4.5 The Ordering of Domain Walls

The cigar moduli space for two domain walls illustrates an important point: domain walls cannot pass each other. In contrast to other solitons, they must satisfy a particular ordering on the line. This is apparent in the moduli space of two domain walls since the relative separation takes values in $R \in \mathbf{R}^+$ rather than $\mathbf{R}$. The picture in spacetime shown in figure 4.

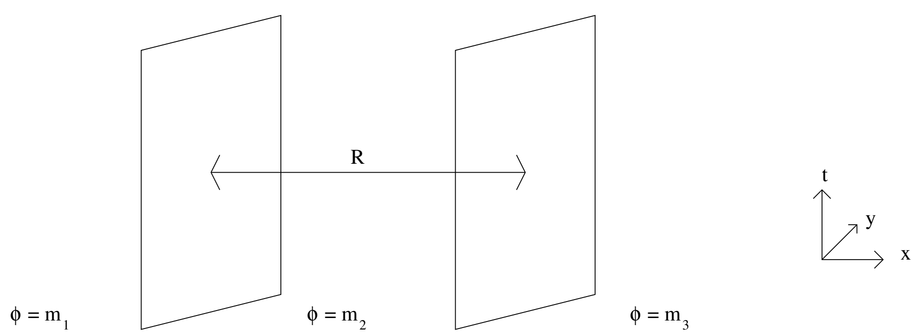

Figure 4: Two interacting domain walls cannot pass through each other. The $\vec{\alpha}_1$ domain wall is always to the left of the $\vec{\alpha}_2$ domain wall.

However, it's not always true that domain walls cannot pass through each other. Domain walls which live in different parts of the flavor group, so have $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$, do not interact so can happily move through each other. When these domain walls are two of many in a topological sector $\vec{g}$, an interesting pattern of interlaced walls arises, determined by which walls bump into each other, and which pass through each other. This pattern was first explored in [9]. Let's see how the ordering emerges. Start at left infinity in a particular vacuum Ξ_-. Then each elementary domain wall shifts the vacuum by increasing a single element $\xi(a) \in \Xi$ by one. The restriction that the N_c elements are distinct means that only certain domain walls can occur. This point is one that is best illustrated by a simple example:

An Example: $N_c = 2$, $N_f = 4$

Consider the domain walls in the $U(2)$ theory with $N_f = 4$ flavors. We'll start at left infinity in the vacuum $\Xi_- = \{1, 2\}$ and end at right infinity in the vacuum $\Xi_+ = \{3, 4\}$. There are two different possibilities for the intermediate vacua. They are:

$$\begin{aligned} \Xi_- = \{1,2\} \longrightarrow \{1,3\} \longrightarrow \{1,4\} \longrightarrow \{2,4\} \longrightarrow \{3,4\} = \Xi_+ \\ \Xi_- = \{1,2\} \longrightarrow \{1,3\} \longrightarrow \{2,3\} \longrightarrow \{2,4\} \longrightarrow \{3,4\} = \Xi_+ \end{aligned}$$

In terms of domain walls, these two ordering become,

$$\begin{aligned} &\vec{\alpha}_2 \longrightarrow \vec{\alpha}_3 \longrightarrow \vec{\alpha}_1 \longrightarrow \vec{\alpha}_2 \\ &\vec{\alpha}_2 \longrightarrow \vec{\alpha}_1 \longrightarrow \vec{\alpha}_3 \longrightarrow \vec{\alpha}_2 \end{aligned} \tag{4.28}$$

We see that the two $\vec{\alpha}_2$ domain walls must play bookends to the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls. However, one expects that these middle two walls are able to pass through each other.

The General Ordering of Domain Walls

We may generalize the discussion above to deduce the rule for ordering of general domain walls [9]. One finds that the n_i elementary $\vec{\alpha}_i$ domain walls must be interlaced between the $\vec{\alpha}_{i-1}$ and $\vec{\alpha}_{i+1}$ domain walls. (Recall that $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$ so the concept of interlacing is well defined). The final pattern of domain walls is captured in figure 5, where x^3 is now plotted horizontally and the vertical position of the domain wall simply denotes its type. We shall see this vertical position take on life in the D-brane set-up we describe shortly.

Notice that the $\vec{\alpha}_1$ domain wall is trapped between the two $\vec{\alpha}_2$ domain walls. These in turn are trapped between the three $\vec{\alpha}_3$ domain walls. However, the relative positions of the $\vec{\alpha}_1$ and middle $\vec{\alpha}_3$ domain walls are not fixed: these objects can pass through each other.

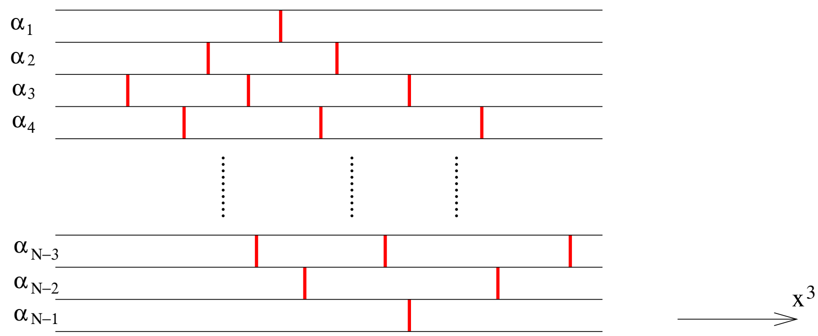

Figure 5: The ordering of many domain walls. The horizontal direction is their position, while the vertical denotes the type of domain wall.

4.6 What Became Of......

Now let's play our favorite game and ask what happened to the other solitons now that we've turned on the masses. We start with.....

4.6.1 Vortices

The vortices described in the previous lecture enjoyed zero modes arising from their embedding in $SU(N)_{\rm diag} \subset U(N_c) \times SU(N_f)$. Let go back to the situation with $N_f = N_c = N$, but with the extra terms from (4.1) added to the Lagrangian,

$$V = \frac{1}{e^2}{\rm Tr}\,(\mathcal{D}_\mu \phi)^2 + \sum_{i=1}^{N} q_i^\dagger (\phi - m_i)^2 q_i \tag{4.29}$$

As we've seen, this mass term breaks $SU(N)_{\rm diag} \rightarrow U(1)^{N-1}_{\rm diag}$, which means we can no longer rotate the orientation of the vortices within the gauge and flavor groups. We learn that the masses are likely to lift some zero modes of the vortex moduli space [22, 23, 24].

The vortex solutions that survive are those whose energy isn't increased by the extra terms in V above. Or, in other words, those vortex configurations which vanish when evaluated on V above. If we don't want the vortex to pick up extra energy from the kinetic terms $\mathcal{D}\phi^2$, we need to keep ϕ in its vacuum,

$$\phi = {\rm diag}(\phi_1, \ldots, \phi_N) \tag{4.30}$$

which means that only the components $q^a{}_i \sim \delta^a{}_i$ can turn on keeping $V = 0$.

For the single vortex $k = 1$ in $U(N)$, this means that the internal moduli space $\mathbb{CP}^{N-1}$ is lifted, leaving behind N different vortex strings, each with magnetic field in a different diagonal component of the gauge group,

$$\begin{aligned} B_3 &= \mathrm{diag}(0, \ldots, B_3^\star, \ldots, 0) \\ q &= \mathrm{diag}(v, \ldots, q^\star, \ldots, v) \end{aligned} \tag{4.31}$$

In summary, rather than having a moduli space of vortex strings, we are left with N different vortex strings, each carrying magnetic flux in a different $U(1) \subset U(N)$.

How do we see this from the perspective of the vortex worldsheet? We can re-derive the vortex theory using the brane construction of the previous lecture, but now with the D6-branes separated in the x^4 direction, providing masses m_i for the hypermultiplets q_i. After performing the relevant brane-game manipulations, we find that these translate into masses m_i for the chiral multiplets in the vortex theory. The potential for the vortex theory (3.30) is replaced by,

$$\begin{aligned} V = &\frac{1}{g^2}\mathrm{Tr}\,|[\sigma, \sigma^\dagger]|^2 + \mathrm{Tr}\,|[\sigma, Z]|^2 + \mathrm{Tr}\,|[\sigma, Z^\dagger]|^2 \\ &+ \sum_{a=1}^{N} \psi_a^\dagger(\sigma - m_a)^2\psi_a + \frac{g^2}{2}\,\mathrm{Tr}\,\left(\sum_a \psi_a\psi_a^\dagger + [Z, Z^\dagger] - r\,1_k\right)^2 \end{aligned} \tag{4.32}$$

where $r = 2\pi/e^2$ as before. The masses m_i of the four-dimensional theory have descended to masses m_a on the vortex worldsheet.

To see the implications of this, consider the theory on a single $k = 1$ vortex. The potential is simply,

$$V_{k=1} = \sum_{a=1}^{N}(\sigma - m_a)^2|\psi_a|^2 + \frac{g^2}{2}(\sum_{a=1}^{N}|\psi_a|^2 - r)^2 \tag{4.33}$$

Whereas before we could set $\sigma = 0$, leaving ψ_a to parameterize $\mathbb{CP}^{N-1}$, now the Higgs branch is lifted. We have instead N isolated vacua,

$$\sigma = m_a \quad , \quad |\psi_b|^2 = r\delta_{ab} \tag{4.34}$$

These correspond to the N different vortex strings we saw above.

A Potential on the Vortex Moduli Space

We can view the masses m_i as inducing a potential on the Higgs branch of the vortex theory after integrating out σ. This potential is equal to the length of Killing vectors on the Higgs branch associated to the $U(1)^{N-1} \subset SU(N)_{\mathrm{diag}}$ isometry. This is the same story we saw in Lecture 2.6, where the a vev for ϕ induced a potential on the instanton moduli space.

In fact, just as we saw for instantons, this result can also be derived directly within the field theory itself [24]. Suppose we fix a vortex configuration (A_z, q) that solves the vortex equations before we introduce masses. We want to determine how much the new terms (4.29) lift the energy of this vortex. We minimize V by solving the equation of motion for ϕ in the background of the vortex,

$$\mathcal{D}^2\phi = \frac{e^2}{2}\sum_{i=1}^{N}\{\phi, q_i q_i^\dagger\} - 2q_i q_i^\dagger m_i \tag{4.35}$$

subject to the vev boundary condition $\phi \to \text{diag}(m_1, \ldots, m_N)$ as $r \to \infty$. But we have seen this equation before! It is precisely the equation (3.21) that an orientational zero mode of the vortex must satisfy. This means that we can write the excess energy of the vortex in terms of the relevant orientational zero mode

$$V = \int d^2x \; \frac{2}{e^2}\text{Tr}\, \delta A_z\, \delta A_{\bar{z}} + \frac{1}{2}\sum_{i=1}^{N} \delta q_i\, \delta q_i^\dagger \tag{4.36}$$

for the particular orientation zero mode $\delta A_z = \mathcal{D}_z\phi$ and $\delta q_i = i(\phi q_i - q_i m_i)$. We can give a nicer geometrical interpretation to this following the discussion in Section 2.6. Denote the Cartan subalgebra of $SU(N)_{\text{diag}}$ as $\vec{H}$, and the associated Killing vectors on $\mathcal{V}_{k,N}$ as $\vec{k}_\alpha$. Then, since ϕ generates the transformation $\vec{m}\cdot\vec{H}$, we can express our zero mode in terms of the basis $\delta A_z = (\vec{m}\cdot\vec{k}^\alpha)\,\delta_\alpha A_z$ and $\delta q_i = (\vec{m}\cdot\vec{k}^\alpha)\,\delta_\alpha q_i$. Putting this in our potential and performing the integral over the zero modes, we have the final expression

$$V = g_{\alpha\beta}\,(\vec{m}\cdot\vec{k}^\alpha)\,(\vec{m}\cdot\vec{k}^\beta) \tag{4.37}$$

This potential vanishes at the fixed points of the $U(1)^{N-1}$ action. For the one-vortex moduli space $\mathbb{CP}^{N-1}$, it's not hard to see that this gives rise to the N vacuum states described above (4.34).

4.6.2 Monopoles

To see where the monopoles have gone, it's best if we first look at the vortex worldsheet theory [22]. This is now have a $d = 1 + 1$ dimensional theory with isolated vacua, guaranteeing the existence of domain wall, or kink, in the worldsheet. In fact, for a single $k = 1$ vortex, the theory on the worldsheet is precisely of the form (4.1) that we started with at the beginning of this lecture. (For $k > 1$, the presence of the adjoint scalar Z means that isn't precisely the same action, but is closely related). The equations describing kinks on the worldsheet are the same as (4.10),

$$\partial_3\sigma = g^2(\sum_{a=1}^{N}|\psi_a|^2 - r) \quad , \quad \mathcal{D}_3\psi_a = (\sigma - m_a)\psi_a \tag{4.38}$$

where we should take the limit $g^2 \to \infty$, in which the first equation becomes algebraic. What's the interpretation of this kink on the worldsheet? We can start by examining its mass,

$$M_{\rm kink} = (\vec{m}\cdot\vec{g})\, r = \frac{2\pi}{e^2}\,(\vec{\phi}\cdot\vec{g}) = M_{\rm mono} \tag{4.39}$$

So the kink has the same mass as the monopole! In fact, it also has the same quantum numbers. To see this, recall that the different vacua on the vortex string correspond to flux tubes lying in different $U(1) \subset U(N)$ subgroups. For example, for $N = 2$, the kink must take the form shown in figure 6. So whatever the kink is, it must soak up magnetic

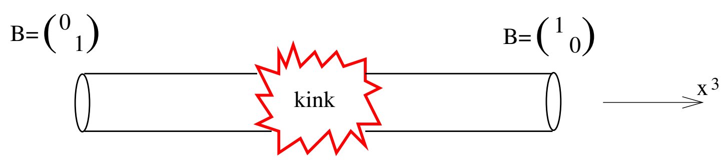

Figure 6: The kink on the vortex string.

field $B = \text{diag}(0, 1)$ and spit out magnetic field $B = \text{diag}(1, 0)$. In other words, it is a source for the magnetic field $B = \text{diag}(1, -1)$. This is precisely the magnetic field sourced by an $SU(2)$ 't Hooft-Polyakov monopole.

What's happening here? We are dealing with a theory with a mass gap, so any magnetic monopole that lives in the bulk can't emit a long-range radial magnetic field since the photon can't propagate. We're witnessing the Meissner effect in a non-abelian superconductor. The monopole is confined, its magnetic field departing in two semi-classical flux tubes. This effect is, of course, well known and it is conjectured that a dual effect leads to the confinement of quarks in QCD. Here we have a simple, semi-classical realization in which to explore this scenario.

Can we find the monopole in the $d = 3+1$ dimensional bulk? Although no solution is known, it turns out that we can write down the Bogomoln'yi equations describing the configuration [22]. Let's go back to our action (4.1) and complete the square in a different way. We now insist only that $\partial_0 = A_0 = 0$, and write the Hamiltonian as,

$$\begin{aligned}\mathcal{H} = \int d^3x\, \frac{1}{e^2}\text{Tr}\, &\left[(\mathcal{D}_1\phi + B_1)^2 + (\mathcal{D}_2 + B_2)^2 + (\mathcal{D}_3\phi + B_3 - \frac{e^2}{2}(\sum_{i=1}^N q_i q_i^\dagger - v^2))^2\right] \\ &+ \sum_{i=1}^N |(\mathcal{D}_1 - i\mathcal{D}_2)q_i|^2 + \sum_{i=1}^N |\mathcal{D}_3 q_i - (\phi - m_i)q_i|^2 + \text{Tr}\,[-v^2 B_3 - \frac{2}{e^2}\partial_i(\phi B_i)]\end{aligned}$$

$$\geq \left(\int dx^3 \; T_{\rm vortex} \right) + M_{\rm mono} \tag{4.40}$$

where the inequality is saturated when the terms in the brackets vanish,

$$\begin{aligned} {\cal D}_1\phi + B_1 = 0 \qquad &, \qquad {\cal D}_1 q_i = i{\cal D}_2 q_i \\ {\cal D}_2\phi + B_2 = 0 \qquad &, \qquad {\cal D}_3 q_i = (\phi - m_i) q_i \\ {\cal D}_3\phi + B_3 = \frac{e^2}{2}(\sum_{i=1}^N q_i q_i^\dagger - v^2) & \end{aligned} \tag{4.41}$$

As you can see, these are an interesting mix of the monopole equations and the vortex equations. In fact, they also include the domain wall equations — we'll see the meaning of this when we come to discuss the applications. These equations should be thought of as the master equations for BPS solitons, reducing to the other equations in various limits. Notice moreover that these equations are over-determined, but it's simple to check that they satisfy the necessary integrability conditions to admit solutions. However, no non-trivial solutions are known analytically. (Recall that even the solution for a single vortex is not known in closed form). We expect that there exist solutions that look like figure 7.

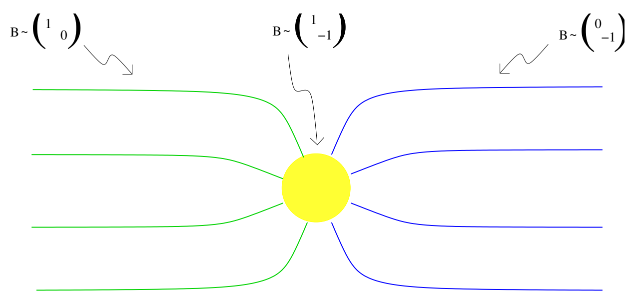

Figure 7: The confined magnetic monopole.

The above discussion was for $k = 1$ and $N_f = N_c$. Extensions to $k \geq 2$ and also to $N_f \geq N_c$ also exist, although the presence of the adjoint scalar Z on the vortex worldvolume means that the kinks on the string aren't quite the same as the domain wall equations (4.10). But if we set $Z = 0$, so that the strings lie on top of each other, then the discussion of domain walls in four-dimensions carries over to kinks on the string. In fact, it's not hard to check that we've chosen our notation wisely: magnetic monopoles of charge $\vec{g}$ descend to kinks on the vortex strings with topological charge $\vec{g}$.

In summary,

Kink on the Vortex String = Confined Magnetic Monopole

The BPS confined monopole was first described in [22], but the idea that kinks on string should be interpreted as confined monopoles arose previously in [25] in the context of Z_N flux tubes. More recently, confined monopoles have been explored in several different theories [26, 27, 28, 29]. We'll devote Section 4.7 to more discussion on this topic.

4.6.3 Instantons

We now ask what became of instantons. At first glance, it doesn't look promising for the instanton! In the bulk, the FI term v^2 breaks the gauge group, causing the instanton to shrink. And the presence of the masses means that even in the center of various solitons, there's only a $U(1)$ restored, not enough to support an instanton. For example, an instanton wishing to nestle within the core of the vortex string shrinks to vanishing size and it looks as if the theory (4.1) admits only singular, small instantons.

While the above paragraph is true, it also tells us how we should change our theory to allow the instantons to return: we should consider non-generic mass parameters, so that the $SU(N_f)$ flavor symmetry isn't broken to the maximal torus, but to some non-abelian subgroup. Let's return to the example discussed in Section 4.5: $U(2)$ gauge theory with $N_f = 4$ flavors. Rather than setting all masses to be different, we chose $m_1 = m_2 = m$ and $m_3 = m_4 = -m$. In this limit, the breaking of the flavor symmetry is $SU(4) \rightarrow S[U(2) \times U(2)]$, and this has interesting consequences.

To find our instantons, we look at the domain wall which interpolates between the two vacua $\phi = m\mathbf{1}_2$ and $\phi = -m\mathbf{1}_2$. When all masses were distinct, this domain wall had 8 collective coordinates which had the interpretation of the position and phase of 4 elementary domain walls (4.28). Now that we have non-generic masses, the domain wall retains all 8 collective coordinates, but some develop a rather different interpretation: they correspond to new orientation modes in the unbroken flavor group. In this way, part of the domain wall theory becomes the $SU(2)$ chiral Lagrangian [30].

Inside the domain wall, the non-abelian gauge symmetry is restored, and the instantons may safely nestle there, finding refuge from the symmetry breaking of the bulk. One can show that, from the perspective of the domain wall worldvolume theory, they appear as Skyrmions [31]. Indeed, closer inspection reveals that the low-energy dynamics of the domain wall also includes a four derivative term necessary to stabilize the Skyrmion, and one can successfully compare the action of the instanton and Skyrmion. The relationship between instantons and Skyrmions was first noted long ago by Atiyah and Manton [32], and has been studied recently in the context of deconstruction [33, 34, 35].

4.7 The Quantum Vortex String

So far our discussion has been entirely classical. Let's now turn to the quantum theory. We have already covered all the necessary material to explain the main result. The basic idea is that $d = 1 + 1$ worldsheet theory on the vortex string captures quantum information about the $d = 3 + 1$ dimensional theory in which it's embedded. If we want certain information about the 4d theory, we can extract it using much simpler calculations in the 2d worldsheet theory.

I won't present all the calculations here, but instead simply give a flavor of the results [23, 24]. The precise relationship here holds for $\mathcal{N} = 2$ theories in $d = 3 + 1$, corresponding to $\mathcal{N} = (2, 2)$ theories on the vortex worldsheet. The first hint that the 2d theory contains some information about the 4d theory in which its embedded comes from looking at the relationship between the 2d FI parameter and the 4d gauge coupling,

$$r = \frac{4\pi}{e^2} \tag{4.42}$$

This is a statement about the classical vortex solution. Both e^2 in 4d and r in 2d run at one-loop. However, the relationship (4.42) is preserved under RG flow since the beta functions computed in 2d and 4d coincide,

$$r(\mu) = r_0 - \frac{N_c}{2\pi} \log\left(\frac{\mu_{UV}}{\mu}\right) \tag{4.43}$$

This ensures that both 4d and 2d theories hit strong coupling at the same scale $\Lambda = \mu \exp(-2\pi r/N_c)$.

Exact results about the 4d theory can be extracted using the Seiberg-Witten solution [36]. In particular, this allows us to determine the spectrum of BPS states in the theory. Similarly, the exact spectrum of the 2d theory can also be determined by computing the twisted superpotential [11, 37]. The punchline is that the spectrum of the two theories coincide. Let's see what this means. We saw in (4.39) that the classical kink mass coincides with the classical monopole mass

$$M_{\text{kink}} = M_{\text{mono}} \tag{4.44}$$

This equality is preserved at the quantum level. Let me stress the meaning of this. The left-hand side is computed in the $d = 1 + 1$ dimensional theory. When $(m_i - m_j) \gg \Lambda$, this theory is weakly coupled and $M_{\rm kink}$ receives a one-loop correction (with, obviously, two-dimensional momenta flowing in the loop). Although supersymmetry forbids higher loop corrections, there are an infinite series of worldsheet instanton contributions. The final expression for the mass of the kink schematically of the form,

$$M = M_{\rm clas} + M_{\rm one-loop} + \sum_{n=1}^{\infty} M_{\rm n-inst} \tag{4.45}$$

The right-hand-side of (4.44) is computed in the $d = 3+1$ dimensional theory, which is also weakly coupled for $(m_i - m_J) \gg \Lambda$. The monopole mass $M_{\rm mono}$ receives corrections at one-loop (now integrating over four-dimensional momenta), followed by an infinite series of Yang-Mills instanton corrections. *And term by term these two series agree*!

The agreement of the worldsheet and Yang-Mills instanton expansions apparently has its microscopic origin in the results if the previous lecture. Recall that performing an instanton computation requires integration over the moduli space ($\mathcal{V}$ for the worldsheet instantons; $\mathcal{I}$ for Yang-Mills). Localization theorems hold when performing the integrals over $\mathcal{I}_{k,N}$ in $\mathcal{N} = 2$ super Yang-Mills, and the final answer contains contributions from only a finite number of points in $\mathcal{I}_{k,N}$ [38]. It is simple to check that all of these points lie on $\mathcal{V}_{k,N}$ which, as we have seen, is a submanifold of $\mathcal{I}_{k,N}$.

The equation (4.44) also holds in strong coupling regimes of the 2d and 4d theories where no perturbative expansion is available. Nevertheless, exact results allow the masses of BPS states to be computed and successfully compared. Moreover, the quantum correspondence between the masses of kinks and monopoles is not the only agreement between the two theories. Other results include:

- The elementary internal excitations of the string can be identified with W-bosons of the 4d theory. When in the bulk, away from the string, these W-bosons are non-BPS. But they can reduce their mass by taking refuge in the core of the vortex whereupon they regain their BPS status.

This highlights an important point: the spectrum of the 4d theory, both for monopoles and W-bosons, is calculated in the Coulomb phase, when the FI parameter $v^2 = 0$. However, the vortex string exists only in the Higgs phase $v^2 \neq 0$. What's going on? A heuristic explanation is as follows: inside the vortex, the Higgs field q dips to zero and the gauge symmetry is restored. The vortex theory captures information about the 4d theory on its Coulomb branch.

- As we saw in Sections 2.3 and 4.4, both the 4d theory and the 2d theory contain dyons. We've already seen that the spectrum of both these objects is given by the "square-root" formula (2.41) and (4.27). Again, these agree at the quantum level.

- Both theories manifest the Witten effect: adding a theta angle to the 4d theory induces an electric charge on the monopole, shifting its mass. This also induces a theta angle on the vortex worldsheet and, hence, turns the kinks into dyons.

- We have here described the theory with $N_f = N_c$. For $N_f > N_C$, the story can be repeated and again the spectrum of the vortex string coincides with the spectrum of the 4d theory in which it's embedded.

In summary, we have known for over 20 years that gauge theories in 4d share many qualitative features with sigma models in 2d, including asymptotic freedom, a dynamically generated mass gap, large N expansions, anomalies and the presences of instantons. However, the vortex string provides a quantitative relationship between the two: in this case, they share the same quantum spectrum.

4.8 The Brane Construction

In Lecture 3, we derived the brane construction for $U(N_c)$ gauge theory with N_f hypermultiplets. To add masses, one must separate the hypermultiplets in the x^4 direction. One can now see the number of vacua (4.6) since each of the N_c D4-branes must end on one of the N_f D6-branes.

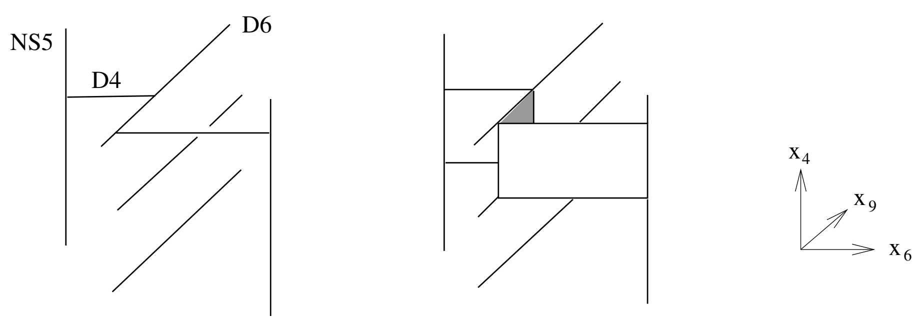

Figure 8: The D-brane configuration for an elementary $\vec{g} = \vec{\alpha}_1$ domain wall when $N_c = 1$ and $N_f = 3$.

To describe a domain wall, the D4-branes must start in one vacua, Ξ_- at $x_3 \to -\infty$, and interpolate to the final vacua Ξ_+ as $x^3 \to +\infty$. Viewing this integrated over all x^3, we have the picture shown in figure 8. To extract the dynamics of domain walls, we need to understand the worldvolume theory of the curved D4-brane. This isn't at all clear. Related issues have troubled previous attempts to extract domain wall dynamics from D-brane set-up [39, 40], although some qualitative features can be seen. However, we can make progress by studying this system in the limit $e^2 \to \infty$, so that the two NS5-branes and the N_f D6-branes lie coincident in the x^6 direction [41]. The portions of the D4-branes stretched in x^6 vanish, and we're left with D4-branes with worldvolume 01249, trapped in squares in the 49 directions where they are sandwiched between the NS5 and D6-branes. Returning to the system of domain walls in an arbitrary topological sector $\vec{g} = \sum_i n_i \vec{\alpha}_i$, we have the system drawn in figure 9.

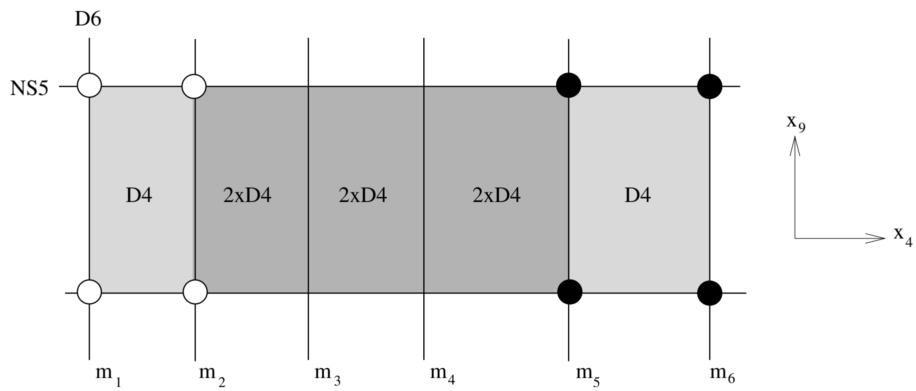

Figure 9: The D-brane configuration in the $e^2 \to \infty$ limit.

We can now read off the gauge theory living on the D4-branes. One might expect that it is of the form $\prod_i U(n_i)$. This is essentially correct. The NS5-branes project out the A_9 component of the gauge field, however the A_4 component survives and each $U(n_a)$ gauge theory lives in the interval $m_i \leq x_4 \leq m_{i+1}$. In each segment, we have A_4 and X_3, each an $n_i \times n_i$ matrix. These fields satisfy

$$\frac{dX_3}{dx^4} - i[A_4, X_3] = 0 \tag{4.46}$$

modulo $U(n_i)$ gauge transformations acting on the interval $m_i \leq x_4 \leq m_{i+1}$, and vanishing at the boundaries. These equations are kind of trivial: the interesting details lie in the boundary conditions. As in the case of monopoles, the interactions between neighbouring segments depends on the relative size of the matrices:

$\underline{n_i = n_{i+1}}$: The $U(n_i)$ gauge symmetry is extended to the interval $m_i \leq x_4 \leq m_{i+2}$ and an impurity is added to the right-hand-side of Nahm's equations, which now read

$$\frac{dX_3}{dx_4} - i[A_4, X_3] = \psi\psi^\dagger \delta(x_4 - m_{i+1}) \tag{4.47}$$

where the impurity degree of freedom ψ transforms in the fundamental representation of the $U(n_i)$ gauge group, ensuring the combination $\psi\psi^\dagger$ is a $n_i \times n_i$ matrix transforming, like X_1, in the adjoint representation. These ψ degrees of freedom are chiral multiplets which survive the NS5-brane projection.

$n_i = n_{i+1} - 1$: In this case $X_3 \rightarrow (X_3)_-$, a $n_i \times n_i$ matrix, as $x_4 \rightarrow (m_i)_-$ from the left. To the right of m_i, X_3 is a $(n_i + 1) \times (n_i + 1)$ matrix obeying

$$X_3 \rightarrow \begin{pmatrix} y & a^\dagger \\ a & (X)_- \end{pmatrix} \qquad \text{as } x_4 \rightarrow (m_i)_+ \tag{4.48}$$

where $y_\mu \in \mathbf{R}$ and each a_μ is a complex n_i-vector. The obvious analog of this boundary condition holds when $n_i = n_{i+1} + 1$.

These boundary conditions are obviously related to the Nahm boundary conditions for monopoles that we met in Lecture 2.

4.8.1 The Ordering of Domain Walls Revisited

We now come to the important point: the ordering of domain walls. Let's see how the brane construction captures this. We can use the gauge transformations to make A_4 constant over the interval $m_i \leq x^4 \leq m_{i+1}$. Then (4.46) can be trivially integrated in each segment to give

$$X_3(x^4) = e^{iA_4 x^4} \hat{X}_3 e^{-iA_4 x^4} \tag{4.49}$$

Then the positions of the $\vec{\alpha}_i$ domain walls are given by the eigenvalues of X_3 restricted to the interval $m_i \leq x_4 \leq m_{i+1}$. Let us denote this matrix as $X_3^{(i)}$ and the eigenvalues as $\lambda_m^{(i)}$, where $m = 1, \ldots n_i$. We have similar notation for the $\vec{\alpha}_{i+1}$ domain walls. Suppose first that $n_i = n_{i+1}$. Then the impurity (4.47) relates the two sets of eigenvalues by the jumping condition

$$X_1^{(i+1)} = X_1^{(i)} + \psi\psi^\dagger \tag{4.50}$$

We will now show that this jumping condition (4.50) correctly captures the interlacing nature of neighboring domain walls.

To see this, consider firstly the situation in which $\psi^\dagger\psi \ll \Delta\lambda_m^{(i)}$ so that the matrix $\psi\psi^\dagger$ may be treated as a small perturbation of $X_1^{(i)}$. The positivity of $\psi\psi^\dagger$ ensures that each $\lambda_m^{(i+1)} \geq \lambda_m^{(i)}$. Moreover, it is simple to show that the $\lambda_m^{(i+1)}$ increase monotonically with $\psi^\dagger\psi$. This leaves us to consider the other extreme, in which $\psi^\dagger\psi \rightarrow \infty$. It this limit ψ becomes one of the eigenvectors of $X_1^{(i+1)}$ with corresponding eigenvalue $\lambda_{n_i}^{(i+1)} = \psi^\dagger\psi$, corresponding to the limit in which the last domain wall is taken to infinity. What we want to show is that the remaining $n_i - 1$ $\vec{\alpha}_{i+1}$ domain walls are trapped between the n_i $\vec{\alpha}_i$ domain walls as depicted in figure 5. Define the $n_i \times n_i$ projection operator

$$P = 1 - \hat{\psi}\hat{\psi}^\dagger \tag{4.51}$$

where $\hat{\psi} = \psi/\sqrt{\psi^\dagger\psi}$. The positions of the remaining $(n_i - 1)$ $\vec{\alpha}_{i+1}$ domain walls are given by the (non-zero) eigenvalues of $PX_1^{(i)}P$. We must show that, given a rank n hermitian matrix X, the eigenvalues of PXP are trapped between the eigenvalues of X. This well known property of hermitian matrices is simple to show:

$$\begin{aligned}\det(PXP - \mu) &= \det(XP - \mu)\\ &= \det(X - \mu - X\hat{\psi}\hat{\psi}^\dagger)\\ &= \det(X - \mu)\det(1 - (X - \mu)^{-1}X\hat{\psi}\hat{\psi}^\dagger)\end{aligned}$$

Since $\hat{\psi}\hat{\psi}^\dagger$ is rank one, we can write this as

$$\begin{aligned}\det(PXP - \mu) &= \det(X - \mu)\left[1 - \mathrm{Tr}((X - \mu)^{-1}X\hat{\psi}\hat{\psi}^\dagger)\right]\\ &= -\mu\,\det(X - \mu)\,\mathrm{Tr}((X - \mu)^{-1}\hat{\psi}\hat{\psi}^\dagger)\\ &= -\mu\left[\prod_{m=1}^{n}(\lambda_m - \mu)\right]\left[\sum_{m=1}^{n}\frac{|\hat{\psi}_m|^2}{\lambda_m - \mu}\right]\end{aligned} \tag{4.52}$$

where $\hat{\psi}_m$ is the m^{th} component of the vector ψ. We learn that PXP has one zero eigenvalue while, if the eigenvalues λ_m of X are distinct, then the eigenvalues of PXP lie at the roots the function

$$R(\mu) = \sum_{m=1}^{n}\frac{|\hat{\psi}_m|^2}{\lambda_m - \mu} \tag{4.53}$$

The roots of $R(\mu)$ indeed lie between the eigenvalues λ_m. This completes the proof that the impurities (4.47) capture the correct ordering of the domain walls.

The same argument shows that the boundary condition (4.48) gives rise to the correct ordering of domain walls when $n_{i+1} = n_i + 1$, with the $\vec{\alpha}_i$ domain walls interlaced between the $\vec{\alpha}_{i+1}$ domains walls. Indeed, it is not hard to show that (4.48) arises from (4.47) in the limit that one of the domain walls is taken to infinity.

4.8.2 The Relationship to Monopoles

You will have noticed that the brane construction above is closely related to the Nahm construction we discussed in Lecture 2. In fact, just as the vortex moduli space $\mathcal{V}_{k,N}$

is related to the instanton moduli space $\mathcal{I}_{k,N}$, so the domain wall moduli space $\mathcal{W}_{\vec{g}}$ is related to the monopole moduli space $\mathcal{M}_{\vec{g}}$. The domain wall theory is roughly a subset of the monopole theory. Correspondingly, the domain wall moduli space is a complex submanifold of the monopole moduli space. To make this more precise, consider the isometry rotating the monopoles in the x^1-x^2 plane (mixed with a suitable $U(1)$ gauge action). It we denote the corresponding Killing vector as h, then

$$\mathcal{W}_{\vec{g}} \cong \mathcal{M}_{\vec{g}}|_{h=0} \tag{4.54}$$

This is the analog of equation (3.36), relating the vortex and instanton moduli spaces.

Nahm's equations have appeared previously in describing domain walls in the $\mathcal{N}=1^\star$ theory [42]. I don't know how those domain walls are related to the ones discussed here.

4.9 Applications

We've already seen one application of kinks in section 4.7, deriving a relationship between 2d sigma models and 4d gauge theories. I'll end with a couple of further interesting applications.

4.9.1 Domain Walls and the 2d Black Hole

Recall that we saw in Section 4.3.2 that the relative moduli space of a two domain walls with charge $\vec{g}=\vec{\alpha}_1+\vec{\alpha}_2$ is the cigar shown in figure 2. Suppose we consider domain walls as strings in a $d=2+1$ dimensional theory, so that the worldvolume of the domain walls is $d=1+1$ dimensional. Then the low-energy dynamics of two domain walls is described by a sigma-model on the cigar.

There is a very famous conformal field theory with a cigar target space. It is known as the two-dimensional black hole [43]. It has metric,

$$ds^2_{BH} = k^2[dR^2 + \tanh^2 R\, d\theta^2] \tag{4.55}$$

The non-trivial curvature at the tip of the cigar is cancelled by a dilaton which has the profile

$$\Phi = \Phi_0 - 2\cosh R \tag{4.56}$$

So is the dynamics of the domain wall system determined by this conformal field theory? Well, not so obviously: the metric on the domain wall moduli space $\mathcal{W}_{\vec{\alpha}_1+\vec{\alpha}_2}$ does not coincide with (4.55). However, $d=1+1$ dimensional theory is not conformal and the metric flows as we move towards the infra-red. There is a subtlety with the dilaton which one can evade by endowing the coordinate R with a suitable anomalous transformation under RG flow. With this caveat, it can be shown that the theory on two domain walls in $d = 2+1$ dimensions does indeed flow towards the conformal theory of the black hole with the identification $k = 2v^2/m$ [44].

The conformal field theory of the 2d black hole is dual to Liouville theory [45, 46]. If we deal with supersymmetric theories, this $\mathcal{N} = (2,2)$ conformal field theory has Lagrangian

$$L_{Liouville} = \int d^4\theta \, \frac{1}{2k}|Y|^2 + \frac{\mu}{2}\int d^2\theta \; e^{-Y} + \text{h.c.} \tag{4.57}$$

and the equivalence between the two theories was proven using the techniques of mirror symmetry in [47]. In fact, one can also prove this duality by studying the dynamics of domain walls. Which is rather cute. We work with the $\mathcal{N} = 4$ (eight supercharges) $U(1)$ gauge theory in $d = 2{+}1$ with N_f charged hypermultiplets. As we sketched above, if we quantize the low-energy dynamics of the domain walls, we find the $\mathcal{N} = (2,2)$ conformal theory on the cigar. However, there is an alternative way to proceed: we could choose first to integrate out some of the matter in three dimensions. Let's get rid of the charged hypermultiplets to leave a low-energy effective action for the vector multiplet. As well as the gauge field, the vector multiplet contains a triplet of real scalars $\boldsymbol{\phi}$, the first of which is identified with the ϕ we met in (4.1). The low-energy dynamics of this effective theory in $d = 2+1$ dimensions can be shown to be

$$L_{eff} = H(\boldsymbol{\phi})\,\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} + H^{-1}(\boldsymbol{\phi})(\partial_\mu\sigma + \boldsymbol{\omega}\cdot\partial_\mu\boldsymbol{\phi})^2 - v^4 H^{-1} \tag{4.58}$$

Here σ is the dual photon (see (2.62)) and $\nabla\times\boldsymbol{\omega} = \nabla H$, while the harmonic function H includes the corrections from integrating out the N_f hypermultiplets,

$$H(\boldsymbol{\phi}) = \frac{1}{e^2} + \sum_{i=1}^{N_f}\frac{1}{|\boldsymbol{\phi} - \mathbf{m}_i|} \tag{4.59}$$

where each triplet $\mathbf{m}_i$ is given by $\mathbf{m}_i = (m_i, 0, 0)$. We can now look for domain walls in this $d = 2{+}1$ effective theory. Since we want to study two domain walls, let's set $N_f = 3$. We see that the theory then has three, isolated vacua, at $\boldsymbol{\phi} = (\phi, 0, 0) = (m_i, 0, 0)$.

We now want to study the domain wall that interpolates between the two outer vacua $\phi = m_1$ and $\phi = m_3$. It's not hard to show that, in contrast to the microscopic theory (4.1), there is no domain wall solutions interpolating between these vacua. One can find a $\vec{\alpha}_1$ domain wall interpolating between $\phi = m_1$ and $\phi = m_2$. There is also a $\vec{\alpha}_2$ domain wall interpolating between $\phi = m_2$ and $\phi = m_3$. But no $\vec{\alpha}_1 + \vec{\alpha}_2$ domain wall between the two extremal vacua $\phi = m_1$ and $\phi = m_3$. The reason is essentially that only a single scalar, ϕ, changes in the domain wall profile, with equations of motion given by flow equations,

$$\partial_3 \phi = v^2 H^{-1}(\phi) \tag{4.60}$$

But since we have only a single scalar field, it must actually pass through the middle vacuum (as opposed to merely getting close) at which point the flow equations tell us $\partial_3 \phi = 0$ and it doesn't move anymore.

Although there is no solution interpolating between $\phi = m_1$ and $\phi = m_2$, one can always write down an approximate solution simply by superposing the $\vec{\alpha}_1$ and $\vec{\alpha}_2$ domain walls in such a way that they are well separated. One can then watch the evolution of this configuration under the equations of motion and, from this, extract an effective force between the domain wall [48]. For the case in hand, this calculation was performed in [44], where it was shown that the force is precisely that arising from the Liouville Lagrangian (4.57). In this way, we can use the dynamics of domain walls to derive the mirror symmetry between the cigar and Liouville theory.

4.9.2 Field Theory D-Branes

As we saw in Section 4.3.2 of this lecture, the moduli space of a single domain wall is $\mathcal{W} \cong \mathbf{R} \times \mathbf{S}^1$. This means that the theory living on the $d = 2+1$ dimensional worldvolume of the domain wall contains a scalar X, corresponding to fluctuations of the domain wall in the x^3 direction, together with a periodic scalar θ determining the phase of the wall. But in $d = 2+1$ dimensions, a periodic scalar can be dualized in favor of a photon living on the wall $4\pi v^2 \partial_\mu \theta = \epsilon_{\mu\nu\rho} F^{\nu\rho}$. Thus the low-energy dynamics of the wall can alternatively be described by a free $U(1)$ gauge theory with a neutral scalar X,

$$L_{\rm wall} = \tfrac{1}{2} T_{\rm wall} \left((\partial_\mu X)^2 + \frac{1}{16\pi^2 v^4} F_{\mu\nu} F^{\mu\nu} \right) \tag{4.61}$$

This is related to the mechanism for gauge field localization described in [49].

As we have seen above, the theory also contains vortex strings. These vortex strings can end on the domain wall, where their ends are electrically charged. In other words, the domain walls are semi-classical D-branes for the vortex strings. These D-branes were first studied in [50, 3, 30]. (Semi-classical D-brane configurations in other theories have been studied in [51, 52, 53] in situations without the worldvolume gauge field). The simplest way to see that the domain wall is D-brane is using the BIon spike described in Section 2.7.4, where we described monopole as D-branes in $d = 5+1$ dimensions.

We can also see this D-brane solution from the perspective of the bulk theory. In fact, the solution obeys the equations (4.41) that we wrote down before. To see this, let's complete the square again but we should be more careful in keeping total derivatives. In a theory with multiple vacua, we have

$$\mathcal{H} = \int d^3x \; \frac{1}{2e^2}\mathrm{Tr}\; \left[(\mathcal{D}_1\phi + B_1)^2 + (\mathcal{D}_2 + B_2)^2 + (\mathcal{D}_3\phi + B_3 - e^2(\sum_{i=1}^{N} q_i q_i^\dagger - v^2))^2\right]$$
$$+\sum_{i=1}^{N} |(\mathcal{D}_1 - i\mathcal{D}_2)q_i|^2 + \sum_{i=1}^{N} |\mathcal{D}_3 q_i - (\phi - m_i)q_i|^2 + \mathrm{Tr}\;[-v^2 B_3 - \frac{1}{e^2}\partial_i(\phi B_i) + v^2\partial_3\phi]$$
$$\geq \left(\int dx^1 dx^2 \; T_{\mathrm{wall}}\right) + \left(\int dx^3 \; T_{\mathrm{vortex}}\right) + M_{\mathrm{mono}} \qquad (4.62)$$

and we indeed find the central charge appropriate for the domain wall. In fact these equations were first discovered in abelian theories to describe D-brane objects [3].

These equations have been solved analytically in the limit $e^2 \to \infty$ [50, 20]. Moreover, when multiple domain walls are placed in parallel along the line, one can construct solutions with many vortex strings stretched between them as figure 10, taken from [20], graphically illustrates.

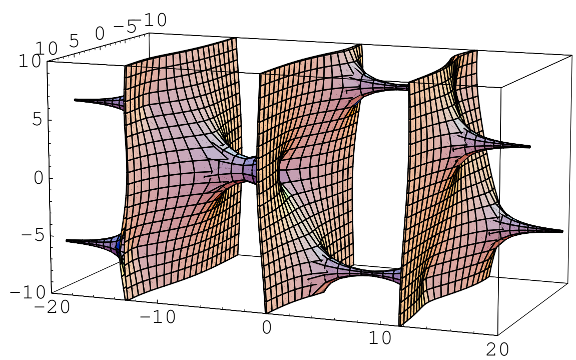

Figure 10: Plot of a field theoretic D-brane configuration [20].

Some final points on the field theoretic D-branes

- In each vacuum there are N_c different vortex strings. Not all of them can end on the bordering domain walls. There exist selection rules describing which vortex string can end on a given wall. For the $\vec{\alpha}_i$ domain wall, the string associated to q_i can end from the left, while the string associated to q_{i+1} can end from the right [6].

- For finite e^2, there is a negative binding energy when the string attaches itself to the domain wall, arising from the monopole central charge in (4.62). Known as a boojum, it was studied in this context in [6, 54]. (The name boojum was given by Mermin to a related configuration in superfluid ^{3}He [55]).

- An interesting question: is the $U(1)$ gauge symmetry on the domain wall worldvolume enhanced as domain walls approach? The answer is: not always. But sometimes! There are new light states that appear as the domain walls approach, but whether these states are associated to scalar particles, giving new light matter, or vector particles, giving gauge symmetry enhancement, depends on the details of the domain wall configuration. Full details will be reported elsewhere [56].

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