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TASI Lectures on Solitons

Lecture 2: Monopoles

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Abstract: In this second lecture we describe the physics of 't Hooft-Polyakov magnetic monopoles when embedded in supersymmetric $SU(N)$ gauge theories. We cover properties of the solutions and the moduli spaces of monopoles and review how Nahm's equations arise in their natural D-brane setting. We end with several applications, including S-duality, the dynamics of three-dimensional gauge theories and field theoretic D-branes.

Contents

2. Monopoles

The tale of magnetic monopoles is well known. They are postulated particles with long-range, radial, magnetic field B_i, $i = 1, 2, 3$,

$$B_i = \frac{g\,\hat{r}_i}{4\pi r^2} \tag{2.1}$$

where g is the magnetic charge. Monopoles have never been observed and one of Maxwell's equations, $\nabla \cdot B = 0$, insists they never will be. Yet they have been a recurrent theme in high energy particle physics for the past 30 years! Why?

The study of monopoles began with Dirac [1] who showed how one could formulate a theory of monopoles consistent with a gauge potential A_μ. The requirement that the electron doesn't see the inevitable singularities in A_μ leads to the famed quantization condition

$$eg = 2\pi n \qquad n \in \mathbf{Z} \tag{2.2}$$

However, the key step in the rehabilitation of magnetic monopoles was the observation of 't Hooft [2] and Polyakov [3] that monopoles naturally occur in non-abelian gauge theories, making them a robust prediction of grand unified theories based on semi-simple groups. In this lecture we'll review the formalism of 't Hooft-Polyakov monopoles in $SU(N)$ gauge groups, including the properties of the solutions and the D-brane realization of the Nahm construction. At the end we'll cover several applications to quantum gauge theories in various dimensions.

There are a number of nice reviews on monopoles in the literature. Aspects of the classical solutions are dealt with by Sutcliffe [4] and Shnir [5]; the mathematics of monopole scattering can be found in the book by Atiyah and Hitchin [6]; the application to S-duality of quantum field theories is covered in the review by Harvey [7]. A comprehensive review of magnetic monopoles by Weinberg and Yi will appear shortly [8].

2.1 The Basics

To find monopoles, we first need to change our theory from that of Lecture 1. We add a single real scalar field $\phi \equiv \phi^a{}_b$, transforming in the adjoint representation of $SU(N)$. The action now reads

$$S = \mathrm{Tr} \int d^4x \; \frac{1}{2e^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu \phi)^2 \tag{2.3}$$

where we're back in Minkowski signature $(+,-,-,-)$. The spacetime index runs over $\mu = 0,1,2,3$ and we'll also use the purely spatial index $i = 1,2,3$. Actions of this type occur naturally as a subsector of $\mathcal{N}=4$ and $\mathcal{N}=2$ super Yang-Mills theories. There is no potential for ϕ so, classically, we are free to chose the vacuum expectation value (vev) as we see fit. Gauge inequivalent choices correspond to different ground states of the theory. By use of a suitable gauge transformation, we may set

$$\langle\phi\rangle = \mathrm{diag}(\phi_1,\ldots,\phi_N) = \vec{\phi}\cdot\vec{H} \tag{2.4}$$

where the fact we're working in $SU(N)$ means that $\sum_{a=1}^N \phi_a = 0$. We've also introduced the notation of the root vector $\vec{\phi}$, with $\vec{H}$ a basis for the $(N-1)$-dimensional Cartan subalgebra of $su(N)$. If you're not familiar with roots of Lie algebras and the Cartn-Weyl basis then you can simply think of $\vec{H}$ as the set of N matrices, each with a single entry 1 along the diagonal. (This is actually the Cartan subalgebra for $u(N)$ rather than $su(N)$ but this will take care of itself if we remember that $\sum_a \phi_a = 0$). Under the requirement that $\phi_a \neq \phi_b$ for $a \neq b$ the gauge symmetry breaks to the maximal torus,

$$SU(N) \rightarrow U(1)^{N-1} \tag{2.5}$$

The spectrum of the theory consists of $(N-1)$ massless photons and scalars, together with $\frac{1}{2}N(N-1)$ massive W-bosons with mass $M_W^2 = (\phi_a - \phi_b)^2$. In the following we will use the Weyl symmetry to order $\phi_a < \phi_{a+1}$.

In the previous lecture, instantons arose from the possibility of winding field configurations non-trivially around the $\mathbf{S}^3_\infty$ infinity of Euclidean spacetime. Today we're interested in particle-like solitons, localized in space rather than spacetime. These objects are supported by the vev (2.4) twisting along its gauge orbit as we circumvent the spatial boundary $\mathbf{S}^2_\infty$. If we denote the two coordinates on $\mathbf{S}^2_\infty$ as θ and φ, then solitons are supported by configurations with $\langle\phi\rangle = \langle\phi(\theta,\varphi)\rangle$. Let's classify the possible windings. A vev of the form (2.4) is one point in a space of gauge equivalent vacua, given by $SU(N)/U(1)^{N-1}$ where the stabilizing group in the denominator is the unbroken symmetry group (2.5) which leaves (2.4) untouched. We're therefore left to consider maps: $\mathbf{S}^2_\infty \rightarrow SU(N)/U(1)^{N-1}$, characterized by

$$\Pi_2\left(SU(N)/U(1)^{N-1}\right) \cong \Pi_1\left(U(1)^{N-1}\right) \cong \mathbf{Z}^{N-1} \tag{2.6}$$

This classification suggests that we should be looking for $(N-1)$ different types of topological objects. As we shall see, these objects are monopoles carrying magnetic charge in each of the $(N-1)$ unbroken abelian gauge fields (2.5).

Why is winding of the scalar field ϕ at infinity associated with magnetic charge? To see the precise connection is actually a little tricky — details can be found in [2, 3] and in [9] for $SU(N)$ monopoles — but there is a simple heuristic argument to see why the two are related. The important point is that if a configuration is to have finite energy, the scalar kinetic term $\mathcal{D}_\mu\phi$ must decay at least as fast as $1/r^2$ as we approach the boundary $r \to \infty$. But if $\langle\phi\rangle$ varies asymptotically as we move around $\mathbf{S}^2_\infty$, we have $\partial\phi \sim 1/r$. To cancel the resulting infrared divergence we must turn on a corresponding gauge potential $A_\theta \sim 1/r$, leading to a magnetic field of the form $B \sim 1/r^2$.

Physically, we would expect any long range magnetic field to propagate through the massless $U(1)$ photons. This is indeed the case. If $\mathcal{D}_i\phi \to 0$ as $r \to \infty$ then $[\mathcal{D}_i, \mathcal{D}_j]\phi = -i[F_{ij}, \phi] \to 0$ as $r \to \infty$. Combining these two facts, we learn that the non-abelian magnetic field carried by the soliton is of the form,

$$B_i = \vec{g} \cdot \vec{H}(\theta, \varphi)\, \frac{\hat{r}_i}{4\pi r^2} \tag{2.7}$$

Here the notation $\vec{H}(\theta, \varphi)$ reminds us that the unbroken Cartan subalgebra twists within the $su(N)$ Lie algebra as we move around the $\mathbf{S}^2_\infty$.

2.1.1 Dirac Quantization Condition

The allowed magnetic charge vectors $\vec{g}$ may be determined by studying the winding of the scalar field ϕ around $\mathbf{S}^2_\infty$. However, since the winding is related to the magnetic charge, and the latter is a characteristic of the long range behavior of the monopole, it's somewhat easier to neglect the non-abelian structure completely and study just the $U(1)$ fields. The equivalence between the two methods is reflected in the equality between first and second homotopy groups in (2.6).

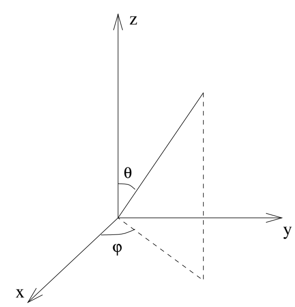

Figure 1:

For this purpose, it is notationally simpler to work in unitary, or singular, gauge in which the vev $\langle\phi\rangle = \vec{\phi} \cdot \vec{H}$ is fixed to be constant at infinity. This necessarily re-introduces Dirac string-like singularities for any single-valued gauge potential, but allows us to globally write the magnetic field in diagonal form,

$$B_i = \text{diag}(g_1, \ldots, g_N)\, \frac{\hat{r}_i}{4\pi r^2} \tag{2.8}$$

where $\sum_{a=1}^N g_a = 0$ since the magnetic field lies in $su(N)$ rather than $u(N)$.

What values of g_a are allowed? A variant of Dirac's original argument, due to Wu and Yang [10], derives the magnetic field (2.8) from two gauge potentials defined respectively on the northern and southern hemispheres of $\mathbf{S}^2_\infty$:

$$\begin{aligned} A^N_\varphi &= \frac{1-\cos\theta}{4\pi r\sin\theta}\,\vec{g}\cdot\vec{H} \\ A^S_\varphi &= -\frac{1+\cos\theta}{4\pi r\sin\theta}\,\vec{g}\cdot\vec{H} \end{aligned} \tag{2.9}$$

where A^N goes bad at the south pole $\theta=\pi$, while A^S sucks at the north pole $\theta=0$. To define a consistent field strength we require that on the overlap $\theta\neq 0,\pi$, the two differ by a gauge transformation which, indeed, they do:

$$A^N_i = U(\partial_i + A^S_i)U^{-1} \tag{2.10}$$

with $U(\theta,\varphi) = \exp(-i\vec{g}\cdot\vec{H}\varphi/2\pi)$. Notice that as we've written it, this relationship only holds in unitary gauge where $\vec{H}$ doesn't depend on θ or φ, requiring that we work in singular gauge. The final requirement is that our gauge transformation is single valued, so $U(\varphi)=U(\varphi+2\pi)$ or, in other words, $\exp(i\,\vec{g}\cdot\vec{H})=1$. This requirement is simply solved by

$$g_a \in 2\pi\mathbf{Z} \tag{2.11}$$

This is the Dirac quantization condition (2.2) in units in which the electric charge $e=1$, a convention which arises from scaling the coupling outside the action in (2.3). In fact, in our theory the W-bosons have charge 2 under any $U(1)$ while matter in the fundamental representation would have charge 1.

There's another notation for the magnetic charge vector $\vec{g}$ that will prove useful. We write

$$\vec{g} = 2\pi\sum_{a=1}^{N-1} n_a\,\vec{\alpha}_a \tag{2.12}$$

where $n_a\in\mathbf{Z}$ by the Dirac quantization condition[1] and $\vec{\alpha}_a$ are the simple roots of $su(N)$. The choice of simple roots is determined by defining $\vec{\phi}$ to lie in a positive Weyl chamber. What this means in practice, with our chosen ordering $\phi_a<\phi_{a+1}$, is that we can write each root as an N-vector, with

$$\begin{aligned} \vec{\alpha}_1 &= (1,-1,0,\ldots,0) \\ \vec{\alpha}_2 &= (0,1,-1,\ldots,0) \end{aligned} \tag{2.13}$$

[1]For monopoles in a general gauge group, the Dirac quantization condition becomes $\vec{g}=4\pi\sum_a n_a\vec{\alpha}^\star_a$ where $\vec{\alpha}^\star_a$ are simple co-roots.

through to

$$\vec{\alpha}_{N-1} = (0, 0, \ldots, 1, -1) \tag{2.14}$$

Then translating between two different notations for the magnetic charge vector we have

$$\begin{aligned} \vec{g} &= \mathrm{diag}(g_1, \ldots, g_N) \\ &= 2\pi\,\mathrm{diag}(n_1\,,\, n_2 - n_1, \ldots, n_{N-1} - n_{N-2}\,,\, -n_{N-1}) \end{aligned} \tag{2.15}$$

The advantage of working with the integers n_a, $a = 1, \ldots, N-1$ will become apparent shortly.

2.1.2 The Monopole Equations

As in Lecture 1, we've learnt that the space of field configurations decomposes into different topological sectors, this time labelled by the vector $\vec{g}$ or, equivalently, the $N-1$ integers n_a. We're now presented with the challenge of finding solutions in the non-trivial sectors. We can again employ a Bogomoln'yi bound argument (this time actually due to Bogomoln'yi [11]) to derive first order equations for the monopoles. We first set $\partial_0 = A_0 = 0$, so we are looking for time independent configurations with vanishing electric field. Then the energy functional of the theory gives us the mass of a magnetic monopole,

$$\begin{aligned} M_{\rm mono} &= \mathrm{Tr}\int d^3x\;\frac{1}{e^2}B_i^2 + \frac{1}{e^2}(\mathcal{D}_i\phi)^2 \\ &= \mathrm{Tr}\int d^3x\;\frac{1}{e^2}\left(B_i \mp \mathcal{D}_i\phi\right)^2 \pm \frac{2}{e^2}\,B_i\mathcal{D}_i\phi \\ &\geq \frac{2}{e^2}\int d^3x\;\partial_i\;\mathrm{Tr}(B_i\phi) \end{aligned} \tag{2.16}$$

where we've used the Bianchi identity $\mathcal{D}_iB_i = 0$ when integrating by parts to get the final line. As in the case of instantons, we've succeeded in bounding the energy by a surface term which measures a topological charge. Comparing with the expressions above we have

$$M_{\rm mono} \geq \frac{|\vec{g}\cdot\vec{\phi}|}{e^2} = \frac{2\pi}{e^2}\sum_{a=1}^{N-1} n_a\phi_a \tag{2.17}$$

with equality if and only if the monopole equations (often called the Bogomoln'yi equations) are obeyed,

$$\begin{aligned} B_i &= \mathcal{D}_i\phi \qquad\quad \text{if } \vec{g}\cdot\vec{\phi} > 0 \\ B_i &= -\mathcal{D}_i\phi \qquad\quad \text{if } \vec{g}\cdot\vec{\phi} < 0 \end{aligned} \tag{2.18}$$

For the rest of this lecture we'll work with $\vec{g}\cdot\vec{\phi} > 0$ and the first of these equations. Our path will be the same as in lecture 1: we'll first examine the simplest solution to these equations and then study its properties before moving on to the most general solutions. So first:

2.1.3 Solutions and Collective Coordinates

The original magnetic monopole described by 't Hooft and Polyakov occurs in $SU(2)$ theory broken to $U(1)$. We have $SU(2)/U(1) \cong \mathbf{S}^2$ and $\Pi_2(\mathbf{S}^2) \cong \mathbf{Z}$. Here we'll describe the simplest such monopole with charge one. To better reveal the topology supporting this monopole (as well as to demonstrate explicitly that the solution is smooth) we'll momentarily revert back to a gauge where the vev winds asymptotically. The solution to the monopole equation (2.18) was found by Prasad and Sommerfield [12]

$$\begin{aligned} \phi &= \frac{\hat{r}_i\sigma^i}{r}\,(vr\coth(vr) - 1) \\ A_\mu &= -\epsilon_{i\mu j}\,\frac{\hat{r}^j\sigma^i}{r}\left(1 - \frac{vr}{\sinh vr}\right) \end{aligned} \tag{2.19}$$

This solution asymptotes to $\langle\phi\rangle = v\sigma^i\hat{r}^i$, where σ^i are the Pauli matrices (i.e. comparing notation with (2.4) in, say, the $\hat{r}^3$ direction, we have $v = -\phi_1 = \phi_2$). The $SU(2)$ solution presented above has 4 collective coordinates, although none of them are written explicitly. Most obviously, there are the three center of mass coordinates. As with instantons, there is a further collective coordinate arising from acting on the soliton with the unbroken gauge symmetry which, in this case, is simply $U(1)$.

For monopoles in $SU(N)$ we can always generate solutions by embedding the configuration (2.19) above into a suitable $SU(2)$ subgroup. Note however that, unlike the situation for instantons, we can't rotate from one $SU(2)$ embedding to another since the $SU(N)$ gauge symmetry is not preserved in the vacuum. Each $SU(2)$ embedding will give rise to a different monopole with different properties — for example, they will have magnetic charges under different $U(1)$ factors.

Of the many inequivalent embeddings of $SU(2)$ into $SU(N)$, there are $(N-1)$ special ones. These have generators given in the Cartan-Weyl basis by $\vec{\alpha}\cdot\vec{H}$ and $E_{\pm\vec{\alpha}}$ where $\vec{\alpha}$ is one of the simple roots (2.13). In a less sophisticated language, these are simply the $(N-1)$ contiguous 2×2 blocks which lie along the diagonal of an $N\times N$ matrix. Embedding the monopole in the a^{th} such block gives rise to the magnetic charge $\vec{g} = \vec{\alpha}_a$.

2.2 The Moduli Space

For a monopole with magnetic charge $\vec{g}$, we want to know how many collective coordinates are contained within the most general solution. The answer was given by E. Weinberg [14]. There are subtleties that don't occur in the instanton calculation, and a variant of the Atiyah-Singer index theorem due to Callias is required [15]. But the result is very simple. Define the moduli space of monopoles with magnetic charge $\vec{g}$ to be $\mathcal{M}_{\vec{g}}$. Then the number of collective coordinates is

$$\dim(\mathcal{M}_{\vec{g}}) = 4\sum_{a=1}^{N-1} n_a \tag{2.20}$$

The interpretation of this is as follows. There exist $(N-1)$ "elementary" monopoles, each associated to a simple root $\vec{\alpha}_a$, carrying magnetic charge under exactly one of the $(N-1)$ surviving $U(1)$ factors of (2.5). Each of these elementary monopoles has 4 collective coordinates. A monopole with general charge $\vec{g}$ can be decomposed into $\sum_a n_a$ elementary monopoles, described by three position coordinates and a phase arising from $U(1)$ gauge rotations.

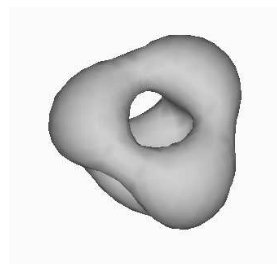

Figure 2:

You should be surprised by the existence of this large class of solutions since it implies that monopoles can be placed at arbitrary separation and feel no force. But this doesn't happen for electrons! Any objects carrying the same charge, whether electric or magnetic, repel. So what's special about monopoles? The point is that monopoles also experience a second long range force due to the massless components of the scalar field ϕ. This gives rise to an attraction between the monopoles that precisely cancels the electromagnetic repulsion [16]. Such cancellation of forces only occurs when there is no potential for ϕ as in (2.3).

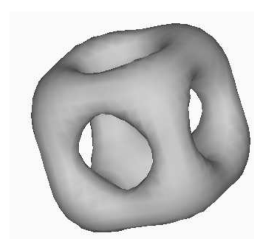

Figure 3:

The interpretation of the collective coordinates as positions of particle-like objects holds only when the monopoles are more widely separated than their core size. As the monopoles approach, weird things happen! Two monopoles form a torus. Three monopoles form a tetrahedron, seemingly splitting into four lumps of energy as seen in figure 2. Four monopoles form a cube as in figure 3. (Both of these figures are taken from [17]). We see that monopoles really lose their individual identities as the approach and merge into each other. Higher monopoles form platonic solids, or buckyball like objects.

2.2.1 The Moduli Space Metric

The metric on $\mathcal{M}_{\vec{g}}$ is defined in a similar fashion to that on the instanton moduli space $\mathcal{I}_{k,N}$. To be more precise, it's defined in an identical fashion. Literally! The key point is that the monopole equations $B = \mathcal{D}\phi$ and the instanton equations $F = {}^\star F$ are really the same: the difference between the two lies in the boundary conditions. To see this, consider instantons with $\partial_4 = 0$ and endow the component of the gauge field $A_4 \equiv \phi$ with a vev $\langle\phi\rangle$. We end up with the monopole equations. So using the notation $\delta\phi = \delta A_4$, we can reuse the linearized self-dual equations (1.14) and the gauge fixing condition (1.17) from the Lecture 1 to define the monopole zero modes. The metric on the monopole moduli space $\mathcal{M}_{\vec{g}}$ is again given by the overlap of zero modes,

$$g_{\alpha\beta} = \frac{1}{e^2}\,\mathrm{Tr}\int d^3x\ (\delta_\alpha A_i\,\delta_\beta A_i + \delta_\alpha\phi\,\delta_\beta\phi) \tag{2.21}$$

The metric on the monopole moduli space has the following properties:

- The metric is hyperKähler .

- The metric enjoys an $SO(3) \times U(1)^{N-1}$ isometry. The former descends from physical rotations of the monopoles in space. The latter arise from the unbroken gauge group. The $U(1)^{N-1}$ isometries are tri-holomorphic, while the $SO(3)$ isometry rotates the three complex structures.

- The metric is smooth. There are no singular points analogous to the small instanton singularities of $\mathcal{I}_{k,N}$ because, as we have seen, the scale of the monopole isn't a collective coordinate. It is fixed to be $L_{\rm mono} \sim 1/M_W$, the Compton wavelength of the W-bosons.

- Since the metrics on $\mathcal{I}_{k,N}$ and $\mathcal{M}_{\vec{g}}$ arise from the same equations, merely endowed with different boundary conditions, one might wonder if we can interpolate between them. In fact we can. In the study of instantons on $\mathbf{R}^3\times\mathbf{S}^1$, with a non-zero Wilson line around the $\mathbf{S}^1$, the $4N$ collective coordinates of the instanton gain the interpretation of the positions of N "fractional instantons" [18, 19]. These are often referred to as calorons and are identified as the monopoles discussed above. By taking the radius of the circle to zero, and some calorons to infinity, we can interpolate between the metrics on $\mathcal{M}_{\vec{g}}$ and $\mathcal{I}_{k,N}$ [20].

2.2.2 The Physical Interpretation of the Metric

For particles such as monopoles in $d = 3+1$ dimensions, the metric on the moduli space has a beautiful physical interpretation first described by Manton [21]. Suppose that the monopoles move slowly through space. We approximate the motion by assuming that the field configurations remain close to the static solutions, but endow the collective coordinates X^α with time dependence: $X^\alpha \rightarrow X^\alpha(t)$. If monopoles collide at very high energies this approximation will not be valid. As the monopoles hit they will spew out massive W-bosons and, on occasion, even monopole-anti-monopole pairs. The resulting field configurations will look nothing like the static monopole solutions. Even for very low-energy scattering it's not completely clear that the approximation is valid since the theory doesn't have a mass gap and the monopoles can emit very soft photons. Nevertheless, there is much evidence that this procedure, known as the *moduli space approximation*, does capture the true physics of monopole scattering at low energies. The time dependence of the fields is

$$A_\mu = A_\mu(X^\alpha(t)) \quad , \qquad \phi = \phi(X^\alpha(t)) \tag{2.22}$$

which reduces the dynamics of an infinite number of field theory degrees of freedom to a finite number of collective coordinates. We must still satisfy Gauss' law,

$$\mathcal{D}_i E_i - i[\phi, \mathcal{D}_0 \phi] = 0 \tag{2.23}$$

which can be achieved by setting $A_0 = \Omega_\alpha \dot{X}^\alpha$, where the Ω_α are the extra gauge rotations that we introduced in (1.15) to ensure that the zero modes satisfy the background gauge fixing condition. This means that the time dependence of the fields is given in terms of the zero modes,

$$\begin{aligned} E_i = F_{0i} &= \delta_\alpha A_i \, \dot{X}^\alpha \\ \mathcal{D}_0 \phi &= \delta_\alpha \phi \, \dot{X}^\alpha \end{aligned} \tag{2.24}$$

Plugging this into the action (2.3) we find

$$\begin{aligned} S &= \mathrm{Tr} \int d^4x \, \frac{1}{e^2} \left(E_i^2 + B_i^2 + (\mathcal{D}_0\phi)^2 + (\mathcal{D}_i \phi)^2 \right) \\ &= \int dt \, \left(M_{\rm mono} + \tfrac{1}{2} g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta \right) \end{aligned} \tag{2.25}$$

The upshot of this is analysis is that the low-energy dynamics of monopoles is given by the $d = 0+1$ sigma model on the monopole moduli space. The equations of motion following from (2.25) are simply the geodesic equations for the metric $g_{\alpha\beta}$. We learn that the moduli space metric captures the velocity-dependent forces felt by monopoles, such that low-energy scattering is given by geodesic motion.

In fact, this logic can be reversed. In certain circumstances it's possible to figure out the trajectories followed by several moving monopoles. From this one can construct a metric on the configuration space of monopoles such that the geodesics reconstruct the known motion. This metric agrees with that defined above in a very different way. This procedure has been carried out for a number of examples [22, 23, 24].

2.2.3 Examples of Monopole Moduli Spaces

Let's now give a few examples of monopole moduli spaces. We start with the simple case of a single monopole where the metric may be explicitly computed.

One Monopole

Consider the $\vec{g} = \vec{\alpha}_1$ monopole, which is nothing more that the charge one $SU(2)$ solution we saw previously (2.19). In this case we can compute the metric directly. We have two different types of collective coordinates:

i) The three translational modes. The linearized monopole equation and gauge fixing equation are solved by $\delta_{(i)} A_j = -F_{ij}$ and $\delta_{(i)}\phi = -\mathcal{D}_i\phi$, so that the overlap of zero modes is

$$\mathrm{Tr}\ \frac{1}{e^2}\int d^3x\ (\delta_{(i)}A_k\,\delta_{(j)}A_k + \delta_{(i)}\phi\,\delta_{(j)}\phi) = M_{\rm mono}\ \delta_{ij} \tag{2.26}$$

ii) The gauge mode arises from transformation $U = \exp(i\phi\chi/v)$, where the normalization has been chosen so that the collective coordinate χ has periodicity 2π. This gauge transformation leaves ϕ untouched while the transformation on the gauge field is $\delta A_i = (\mathcal{D}_i\phi)/v$.

Putting these two together, we find that single monopole moduli space is

$$\mathcal{M}_{\vec{\alpha}} \cong \mathbf{R}^3 \times \mathbf{S}^1 \tag{2.27}$$

with metric

$$ds^2 = M_{\rm mono}\ \left(dX^i dX^i + \frac{1}{v^2}\,d\chi^2\right) \tag{2.28}$$

where $M_{\rm mono} = 4\pi v/e^2$ in the notation used in the solution (2.19).

Two Monopoles

Two monopoles in $SU(2)$ have magnetic charge $\vec{g} = 2\alpha_1$. The direct approach to compute the metric that we have just described becomes impossible since the most general analytic solution for the two monopole configuration is not available. Nonetheless, Atiyah and Hitchin were able to determine the two monopole moduli space using symmetry considerations alone, most notably the constraints imposed by hyperKählerity [6, 25]. It is

$$\mathcal{M}_{2\vec{\alpha}} \cong \mathbf{R}^3 \times \frac{\mathbf{S}^1 \times \mathcal{M}_{AH}}{\mathbf{Z}_2} \tag{2.29}$$

where $\mathbf{R}^3$ describes the center of mass of the pair of monopoles, while $\mathbf{S}^1$ determines the overall phase $0 \leq \chi \leq 2\pi$. The four-dimensional hyperKähler space $\mathcal{M}_{AH}$ is the famous Atiyah-Hitchin manifold. Its metric can be written as

$$ds^2 = f(r)^2 dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2 \tag{2.30}$$

Here the radial coordinate r measures the separation between the monopoles in units of the monopole mass. The σ_i are the three left-invariant $SU(2)$ one-forms which, in terms of polar angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$, take the form

$$\begin{aligned} \sigma_1 &= -\sin\psi \ d\theta + \cos\psi \sin\theta \ d\phi \\ \sigma_2 &= \cos\psi \ d\theta + \sin\psi \sin\theta \ d\phi \\ \sigma_3 &= d\psi + \cos\theta \ d\phi \end{aligned} \tag{2.31}$$

For far separated monopoles, θ and ϕ determine the angular separation while ψ is the relative phase. The $\mathbf{Z}_2$ quotient in (2.29) acts as

$$\mathbf{Z}_2 : \chi \rightarrow \chi + \pi \quad , \quad \psi \rightarrow \psi + \pi \tag{2.32}$$

The hyperKähler condition can be shown to relate the four functions f, a, b and c through the differential equation

$$\frac{2bc}{f}\frac{da}{dr} = (b-c)^2 - a^2 \tag{2.33}$$

together with two further equations obtained by cyclically permuting a, b and c. The solutions can be obtained in terms of elliptic integrals but it will prove more illuminating to present the asymptotic expansion of these functions. Choosing coordinates such that

$f(r) = -b(r)/r$,

$$\begin{aligned} a^2 &= r^2\left(1-\frac{2}{r}\right) - 8r^3e^{-r} + \ldots \\ b^2 &= r^2\left(1-\frac{2}{r}\right) + 8r^3e^{-r} + \ldots \\ c^2 &= 4\left(1-\frac{2}{r}\right)^{-1} + \ldots \end{aligned} \tag{2.34}$$

If we suppress the exponential corrections, the metric describes the velocity dependant forces between two monopoles interacting through their long range fields. In fact, this asymptotic metric can be derived by treating the monopoles as point particles and considering their Liénard-Wiechert potentials. Note that in this limit there is an isometry associated to the relative phase ψ. However, the minus sign before the $2/r$ terms means that the metric is singular. The exponential corrections above contain the information about the behavior of the monopoles as their non-abelian cores overlap, and the full metric is smooth.

The Atiyah-Hitchin metric appears in several places in string theory and supersymmetric gauge theories, including the M-theory lift of the type IIA O6-plane [26], the solution of the quantum dynamics of 3d gauge theories [27], in intersecting brane configurations [28], the heterotic string compactified on ALE spaces [29, 30] and NS5-branes on orientifold 8-planes [31]. In each of these places, there is often a relationship to magnetic monopoles underlying the appearance of this metric.

For higher charge monopoles of the same type $\vec{g} = n\vec{\alpha}$, the leading order terms in the asymptotic expansion of the metric, associated with the long-range fields of the monopoles, have been computed. The result is known as the Gibbons-Manton metric [23]. The full metric on the monopole moduli space remains an open problem.

Two Monopoles of Different Types

As we have seen, higher rank gauge groups $SU(N)$ for $N \geq 3$ admit monopoles of different types. If a $\vec{g} = \vec{\alpha}_a$ monopole and a $\vec{g} = \vec{\alpha}_b$ monopole live in entirely different places in the gauge group, so that $\vec{\alpha}_a \cdot \vec{\alpha}_b = 0$, then they don't see each other and their moduli space is simply the product $(\mathbf{R}^3 \times \mathbf{S}^1)^2$. However, if they live in neighboring subgroups so that $\vec{\alpha}_a \cdot \vec{\alpha}_b = -1$, then they do interact non-trivially.

The metric on the moduli space of two neighboring monopoles, sometimes referred to as the $(1, 1)$ monopole, was first computed by Connell [32]. But he chose not to publish. It was rediscovered some years later by two groups when the connection with electro-magnetic duality made the study of monopoles more pressing [33, 34]. It is simplest to describe if the two monopoles have the same mass, so $\vec{\phi}\cdot\vec{\alpha}_a = \vec{\phi}\cdot\vec{\alpha}_b$. The moduli space is then

$$\mathcal{M}_{\vec{\alpha}_1+\vec{\alpha}_2} \cong \mathbf{R}^3 \times \frac{\mathbf{S}^1 \times \mathcal{M}_{TN}}{\mathbf{Z}_2} \tag{2.35}$$

where the interpretation of the $\mathbf{R}^3$ factor and $\mathbf{S}^1$ factor are the same as before. The relative moduli space is the Taub-NUT manifold, which has metric

$$ds^2 = \left(1+\frac{2}{r}\right)(dr^2 + r^2(\sigma_1^2+\sigma_2^2)) + \left(1+\frac{2}{r}\right)^{-1}\sigma_3^2 \tag{2.36}$$

The $+2/r$ in the metric, rather than the $-2/r$ of Atiyah-Hitchin, means that the metric is smooth. The apparent singularity at $r=0$ is merely a coordinate artifact, as you can check by transforming to the variables $R=\sqrt{r}$. Once again, this $1/r$ terms capture the long range interactions of the monopoles, with the minus sign traced to the fact that each sees the other with opposite magnetic charge (essentially because $\vec{\alpha}_1\cdot\vec{\alpha}_2=-1$). There are no exponential corrections to this metric. The non-abelian cores of the two monopoles do not interact.

The exact moduli space metric for a string of neighboring monopoles, $\vec{g}=\sum_a \vec{\alpha}_a$ has been determined. Known as the Lee-Weinberg-Yi metric, it is a higher dimensional generalization of the Taub-NUT metric [24]. It is smooth and has no exponential corrections.

2.3 Dyons

Consider the one-monopole moduli space $\mathbf{R}^3\times\mathbf{S}^1$. Motion in $\mathbf{R}^3$ is obvious. But what does motion along the $\mathbf{S}^1$ correspond to?

We can answer this by returning to our specific $SU(2)$ solution (2.19). We determined that the zero mode for the $U(1)$ action is $\delta A_i = \mathcal{D}_i\phi$ and $\delta\phi = 0$. Translating to the time dependence of the fields (2.24), we find

$$E_i = \frac{(\mathcal{D}_i\phi)}{v}\,\dot{\chi} = \frac{B_i}{v}\;e^2\dot{\chi} \tag{2.37}$$

We see that motion along the $\mathbf{S}^1$ induces an electric field for the monopole, proportional to its magnetic field. In the unbroken $U(1)$, this gives rise to a long range electric field,

$$\mathrm{Tr}(E_i\phi) = \frac{qve^2\hat{r}_i}{2\pi r^2} \tag{2.38}$$

where, comparing with the normalization above, the electric charge q is given by

$$q = \frac{2\pi\dot{\chi}}{ve^2} \tag{2.39}$$

Note that motion in $\mathbf{R}^3$ also gives rise to an electric field, but this is the dual to the familiar statement that a moving electric charge produces a magnetic field. Motion in $\mathbf{S}^1$, on the other hand, only has the effect of producing an electric field [35].

A particle with both electric and magnetic charges is called a *dyon*, a term first coined by Schwinger [36]. Since we have understood this property from the perspective of the monopole worldline, can we return to our original theory (2.3) and find the corresponding solution there? The answer is yes. We relax the condition $E_i = 0$ when completing the Bogomoln'yi square in (2.16) and write

$$\begin{aligned} M_{\rm dyon} = \mathrm{Tr} \int d^3x \; \frac{1}{e^2}\left(E_i - \cos\alpha\,\mathcal{D}_i\phi\right)^2 + \frac{1}{e^2}\left(B_i - \sin\alpha\,\mathcal{D}_i\phi\right)^2 \\ +\frac{2}{e^2}\mathrm{Tr}\int d^3x\; \partial_i\left(\cos\alpha\, E_i\phi + \sin\alpha\, B_i\phi\right) \end{aligned} \tag{2.40}$$

which holds for all α. We write the long range magnetic field as $E_i = \vec{q}\cdot\vec{H}\, e^2\hat{r}^i/4\pi r^2$. Then by adjusting α to make the bound as tight as possible, we have

$$M_{\rm dyon} \geq \sqrt{\left(\vec{q}\cdot\vec{\phi}\right)^2 + \left(\frac{\vec{g}\cdot\vec{\phi}}{e^2}\right)^2} \tag{2.41}$$

and, given a solution to the monopole, it is easy to find a corresponding solution for the dyon for which this bound is saturated, with the fields satisfying

$$B_i = \sin\alpha\,\mathcal{D}_i\phi \quad \text{and} \quad E_i = \cos\alpha\,\mathcal{D}_i\phi \tag{2.42}$$

This method of finding solutions in the worldvolume theory of a soliton, and subsequently finding corresponding solutions in the parent 4d theory, will be something we'll see several more times in later sections.

I have two further comments on dyons.

- We could add a theta term $\theta F\wedge F$ to the 4d theory. Careful calculation of the electric Noether charges shows that this induces an electric charge $\vec{q} = \theta\vec{g}/2\pi$ on the monopole. In the presence of the theta term, monopoles become dyons. This is known as the Witten effect [37].

- Both the dyons arising from (2.42), and those arising from the Witten effect, have $\vec{q} \sim \vec{g}$. One can create dyons whose electric charge vector is not parallel to the magnetic charge by turning on a vev for a second, adjoint scalar field [38, 39]. These states are 1/4-BPS in $\mathcal{N} = 4$ super Yang-Mills and correspond to (p, q)-string webs stretched between D3-branes. From the field theory perspective, the dynamics of these dyons is described by motion on the monopole moduli space with a potential induced by the second scalar vev [40, 41, 42].

2.4 Fermi Zero Modes

As with instantons, when the theory includes fermions they may be turned on in the background of the monopole without raising the energy of the configuration. A Dirac fermion λ in the adjoint representation satisfies

$$i\gamma^\mu \mathcal{D}_\mu \lambda - i[\phi, \lambda] = 0 \tag{2.43}$$

Each such fermion carried $4\sum_a n_a$ zero modes.

Rather than describing this in detail, we can instead resort again to supersymmetry. In $\mathcal{N} = 4$ super Yang-Mills, the monopoles preserve one-half the supersymmetry, corresponding to $\mathcal{N} = (4, 4)$ on the monopole worldvolume. While, monopoles in $\mathcal{N} = 2$ supersymmetric theories preserve $\mathcal{N} = (0, 4)$ on their worldvolume. Monopoles in $\mathcal{N} = 1$ theories are not BPS; they preserve no supersymmetry on their worldvolume.

There is also an interesting story with fermions in the fundamental representation, leading to the phenomenon of solitons carrying fractional fermion number [43]. A nice description of this can be found in [7].

2.5 Nahm's Equations

In the previous section we saw that the ADHM construction gave a powerful method for understanding instantons, and that it was useful to view this from the perspective of D-branes in string theory. You'll be pleased to learn that there exists a related method for studying monopoles. It's known as the Nahm construction [44]. It was further developed for arbitrary classical gauge group in [45], while the presentation in terms of D-branes was given by Diaconescu [46].

We start with $\mathcal{N} = 4$ $U(N)$ super Yang-Mills, realized on the worldvolume of D3-branes. To reflect the vev $\langle\phi\rangle = \text{diag}(\phi_1 \ldots, \phi_N)$, we separate the D3-branes in a transverse direction, say the x^6 direction. The a^{th} D3-brane is placed at position $x_6 = \phi_a$.

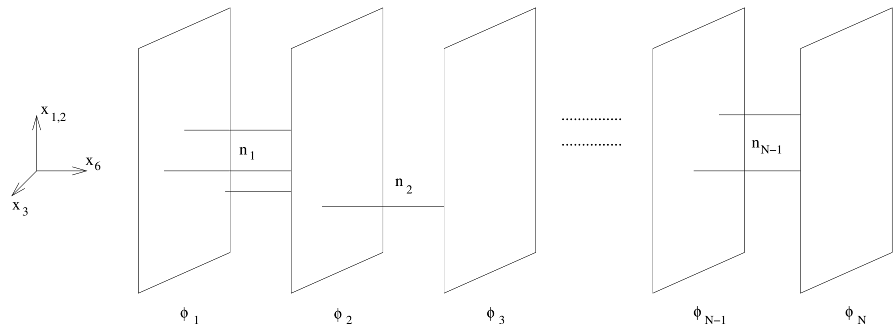

Figure 4: The D-brane set-up for monopoles of charge $\vec{g} = \sum_a n_a \vec{\alpha}_a$.

As is well known, the W-bosons correspond to fundamental strings stretched between the D3-branes. The monopoles are their magnetic duals, the D-strings. At this point our notation for the magnetic charge vector $\vec{g} = \sum_a n_a \vec{\alpha}_a$ becomes more visual. This monopole in sector $\vec{g}$ is depicted by stretching n_a D-strings between the $a^{\rm th}$ and $(a+1)^{\rm th}$ D3-branes.

Our task now is to repeat the analysis of lecture 1 that led to the ADHM construction: we must read off the theory on the D1-branes, which we expect give us a new perspective on the dynamics of magnetic monopoles. From the picture it looks like the dynamics of the D-strings will be governed by something like a $\prod_a U(n_a)$ gauge theory, with each group living on the interval $\phi_a \leq x_6 \leq \phi_{a+1}$. And this is essentially correct. But what are the relevant equations dictating the dynamics? And what happens at the boundaries?

To get some insight into this, let's start by considering n infinite D-strings, with worldvolume x_0, x_6, and with D3-brane impurities inserted at particular points $x_6 = \phi_a$, as shown below.

The theory on the D-strings is a $d = 1+1$ $U(n)$ gauge theory with 16 supercharges (known as $\mathcal{N} = (8,8)$). Each D3-brane impurity donates a hypermultiplet to the theory, breaking supersymmetry by half to $\mathcal{N} = (4,4)$. As in lecture 1, we write the hypermultiplets as

$$\omega_a = \begin{pmatrix} \psi_a \\ \tilde{\psi}_a^\dagger \end{pmatrix} \qquad a = 1, \ldots, N \tag{2.44}$$

where ψ_a transforms in the $\mathbf{n}$ of $U(n)$, while $\tilde{\psi}_a$ transforms in the $\bar{\mathbf{n}}$. The coupling of these impurities (or defects as they're also known) is uniquely determined by supersymmetry, and again occurs in a triplet of D-terms (or, equivalently, a D-term and an

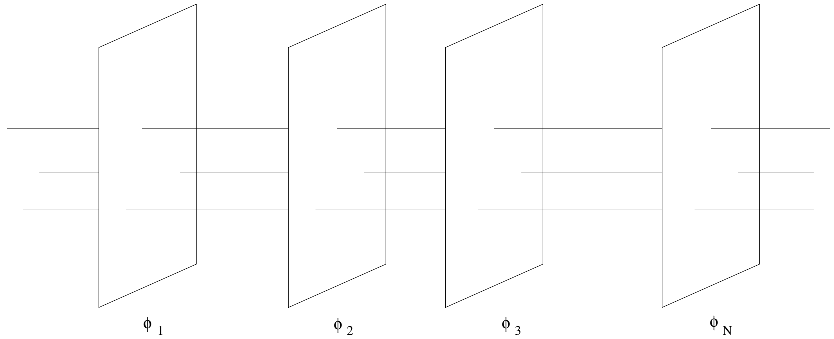

Figure 5: The D3-branes give rise to impurities on the worldvolume of the D1-branes.

F-term). In lecture 1, I unapologetically quoted the D-term and F-term arising in the ADHM construction (equation (1.44)) since they can be found in any supersymmetry text book. However, now we have an impurity theory which is a little less familiar. Nonetheless, I'm still going to quote the result, but this time I'll apologize. We could derive this interaction by examining the supersymmetry in more detail, but it's easier to simply tell you the answer and then give a couple of remarks to try and convince you that it's right. It turns out that the (admittedly rather strange) triplet of D-terms occurring in the Lagrangian is

$$\mathrm{Tr}\,\left(\frac{\partial X^i}{\partial x^6} - i[A_6, X^i] - \frac{i}{2}\epsilon_{ijk}[X^j, X^k] + \sum_{a=1}^{N} \omega_a^\dagger \sigma^i \omega_a\ \delta(x^6 - \phi_a)\right)^2 \tag{2.45}$$

In the ground state of the D-strings, this term must vanish. Some motivating remarks:

- The configuration shown in figure 5 arises from T-dualizing the D0-D4 system. This viewpoint makes it clear that A_6 is the right bosonic field to partner X^i in a hypermultiplet.
- Set $\partial_6 = 0$. Then, relabelling $A_6 = X^4$, this term is almost the same as the triplet of D-terms appearing in the ADHM construction. The only difference is the appearance of the delta-functions.
- We know that D-strings can end on D3-branes. The delta-function sources in the D-term are what allow this to happen. For example, consider a single $n = 1$ D-string, so that all commutators above vanish. We choose $\tilde{\psi} = 0$, to find the triplet of D-terms

$$\partial_6 X^1 = 0 \quad , \quad \partial_6 X^2 = 0 \quad , \quad \partial_6 X^3 = |\psi|^2 \delta(0) \tag{2.46}$$

which allows the D-string profile to take the necessary step (function) to split on the D3-brane as shown below.

If that wasn't enough motivation, one can find the full supersymmetry analysis in the original papers [47, 48] and, in most detail, in [49]. Accepting (2.45) we can make progress in understanding monopole dynamics by studying the limit in which several D-string segments, including the semi-infinite end segments, move off to infinity, leaving us back with the picture of figure 4.

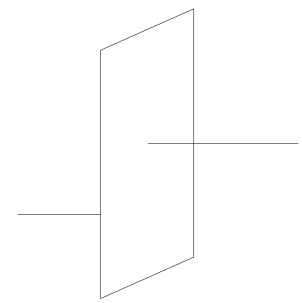

Figure 6:

The upshot of this is that the dynamics of the $\vec{g} = \sum_a n_a \vec{\alpha}_a$ monopoles are described as follows: In the interval $\phi_a \leq x_6 \leq \phi_{a+1}$, we have a $U(n_a)$ gauge theory, with three adjoint scalars X^i, $i = 1, 2, 3$ satisfying

$$\frac{dX^i}{dx^6} - i[A_6, X^i] - \tfrac{1}{2}\epsilon^{ijk}[X^i, X^j] = 0 \tag{2.47}$$

These are Nahm's equations. The boundary conditions imposed at the end of the interval depend on the number of monopoles in the neighbouring segment. (Set $n_0 = n_N = 0$ in what follows)

$\underline{n_a = n_{a+1}}$: The $U(n_a)$ gauge symmetry is extended to the interval $\phi_a \leq x^6 \leq \phi_{a+2}$ and an impurity is added to the right-hand-side of Nahm's equations

$$\frac{dX^i}{dx^6} - i[A_6, X^i] - \tfrac{1}{2}\epsilon^{ijk}[X^i, X^j] = \omega^\dagger_{a+1}\sigma^i\omega_{a+1}\ \delta(x^6 - \phi_{a+1}) \tag{2.48}$$

This, of course, follows immediately from (2.45).

$\underline{n_a = n_{a+1} - 1}$: In this case, $X^i \to (X_i)_-$, a set of three constant $n_a \times n_a$ matrices as $x^6 \to (\phi_{a+1})_-$. To the right of the impurity, the X^i are $(n_a + 1) \times (n_a + 1)$ matrices. They are required to satisfy the boundary condition

$$X^i \to \begin{pmatrix} y^i & a^{i\dagger} \\ a^i & (X^i)_- \end{pmatrix} \qquad \text{as } x_6 \to (\phi_{a+1})_+ \tag{2.49}$$

where $y^i \in \mathbf{R}$ and each a^i is a complex n_a-vector. One can derive this boundary condition without too much trouble by starting with (2.48) and taking $|\omega| \to \infty$ to remove one of the monopoles [50].

$\underline{n_a \leq n_{a+1} - 2}$ Once again $X^i \to (X^i)_-$ as $x_6 \to (\phi_{a+1})_-$ but, from the other side, the matrices X_μ now have a simple pole at the boundary,

$$X^i \to \begin{pmatrix} J^i/s + Y^i & \mathcal{O}(s^\gamma) \\ \mathcal{O}(s^\gamma) & (X^i)_- \end{pmatrix} \qquad \text{as } x_6 \to (\phi_{a+1})_+ \tag{2.50}$$

Here $s = (x^6 - \phi_{a+1})$ is the distance to the impurity. The matrices J^i are the irreducible $(n_{a+1} - n_a) \times (n_{a+1} - n_a)$ representation of $su(2)$, and Y^i are now constant $(n_{a+1} - n_a) \times (n_{a+1} - n_a)$ matrices. Note that the simple pole structure is compatible with Nahm's equations, with both the derivative and the commutator term going like $1/s^2$. Finally, $\gamma = \frac{1}{2}(n_{a+1} - n_a - 1)$, so the off-diagonal terms vanish as we approach the boundary. The boundary condition (2.50) can also be derived from (2.49) by removing a monopole to infinity [50] *except* for the requirement that J^i is irreducible. And this is important! Without this restriction, one can't even build a solution in the right gauge group. As far as I'm aware, it's an open problem to derive the irreducibility of the J^i in the D-brane language.

When $n_a > n_{a+1}$, the obvious parity flipped version of the above conditions holds.

2.5.1 Constructing the Solutions

Just as in the case of ADHM construction, Nahm's equations capture information about both the monopole solutions and the monopole moduli space. The space of solutions to Nahm's equations (2.47), subject to the boundary conditions detailed above, is isomorphic to the monopole moduli space $\mathcal{M}_{\vec{g}}$. The phases of each monopole arise from the gauge field A_6, while X^i carry the information about the positions of the monopoles. Moreover, there is a natural metric on the solutions to Nahm's equations which coincides with the metric on the monopole moduli space. I don't know if anyone has calculated the Atiyah-Hitchin metric using Nahm data, but a derivation of the Lee-Weinberg-Yi metric was given in [51].

Given a solution to Nahm's equations, one can explicitly construct the corresponding solution to the monopole equation. The procedure is analogous to the construction of instantons in 1.4.2, although its a little harder in practice as its not entirely algebraic. We now explain how to do this. The first step is to build a Dirac-like operator from the solution to (2.47). In the segment $\phi_a \leq x^6 \leq \phi_{a+1}$, we construct the Dirac operator

$$\Delta = \frac{d}{dx^6} - iA_6 - i(X^i + r^i)\sigma^i \tag{2.51}$$

where we've reintroduced the spatial coordinates r^i into the game. We then look for normalizable zero modes U which are solutions to the equation

$$\Delta U = 0 \tag{2.52}$$

One can show that there are N such solutions, and so we consider U as a $2n_a \times N$-dimensional matrix. Note that the dimension of U jumps as we move from one interval to the next. We want to appropriately normalize U, and to do so choose to integrate over all intervals, so that

$$\int_{\phi_1}^{\phi_N} dx^6 \; U^\dagger U = \mathbf{1}_N \tag{2.53}$$

Once we've figured out the expression for U, a Higgs field ϕ and a gauge field A_i which satisfy the monopole equation are given by,

$$\phi = \int_{\phi_1}^{\phi_N} dx^6 \; x^6 \, U^\dagger U \quad , \quad A_i = \int_{\phi_1}^{\phi_N} dx^6 \; U^\dagger \partial_6 U \tag{2.54}$$

The similarity between this procedure and that described in section 1.4.2 for instantons should be apparent.

In fact, there's a slight complication that I've brushed under the rug. The above construction only really holds when $n_a \neq n_{a+1}$. If we're in a situation where $n_a = n_{a+1}$ for some a, then we have to take the hypermultiplets ω_a into account, since their value affects the monopole solution. This isn't too hard — it just amounts to adding some extra discrete pieces to the Dirac operator Δ. Details can be found in [45].

A string theory derivation of the construction part of the Nahm construction was recently given in [52].

An Example: The Single $SU(2)$ Monopole Revisited

It's very cute to see the single $n = 1$ solution (2.19) for the $SU(2)$ monopole drop out of this construction. This is especially true since the Nahm data is trivial in this case: $X^i = A_6 = 0$!

To see how this arises, we look for solutions to

$$\Delta U = \left(\frac{d}{dx^6} - r^i \sigma^i \right) U = 0 \tag{2.55}$$

where $U = U(x^6)$ is a 2×2 matrix. This is trivially solved by

$$U = \sqrt{\frac{r}{\sinh(2vr)}} \left(\cosh(rx^6)\, \mathbf{1}_2 + \sinh(rx^6)\, \hat{r}^i \sigma^i \right) \tag{2.56}$$

which has been designed to satisfy the normalizability condition $\int_{-v}^{+v} U^\dagger U \; dx^6 = \mathbf{1}_2$. Armed with this, we can easily reproduce the monopole solution (2.19). For example, the Higgs field is given by

$$\phi = \int_{-v}^{+v} dx^6 \; x^6 U^\dagger U = \frac{\hat{r}^i \sigma^i}{r} \left(vr \coth(vr) - 1 \right) \tag{2.57}$$

as promised. And the gauge field A_i drops out just as easily. See — told you it was cute! Monopole solutions with charge of the type $(1, 1, \ldots, 1)$ were constructed using this method in [53].

2.6 What Became of Instantons

In the last lecture we saw that pure Yang-Mills theory contains instanton solutions. Now we've added a scalar field, where have they gone?! The key point to note is that the theory was conformal before ϕ gained its vev. As we saw in Lecture 1, this led to a collective coordinate ρ, the scale size of the instanton. Now with $\langle \phi \rangle \neq 0$ we have introduced a mass scale into the game and no longer expect ρ to correspond to an exact collective coordinate. This turns out to be true: in the presence of a non-zero vev $\langle \phi \rangle$, the instanton minimizes its action by shrinking to zero size $\rho \to 0$. Although, strictly speaking, no instanton exists in the theory with $\langle \phi \rangle = 0$, they still play a crucial role. For example, the famed Seiberg-Witten solution can be thought of as summing these small instanton corrections.

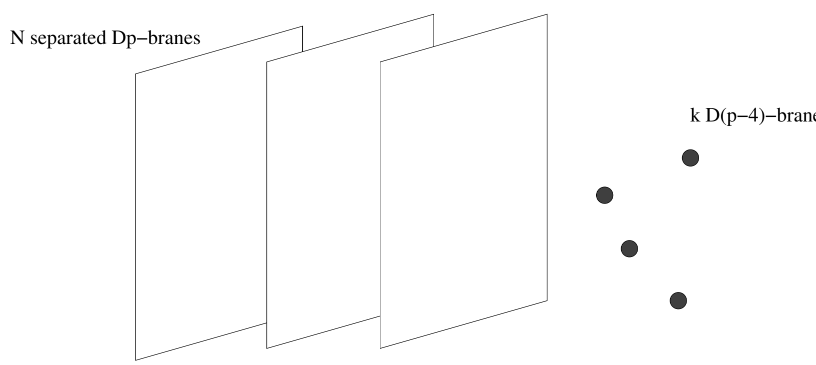

Figure 7: Separating the Dp-branes gives rise to a mass for the hypermultiplets

How can we see this behavior from the perspective of the worldvolume theory? We can return to the D-brane set-up, now with the Dp-branes separated in one direction, say x_6, to mimic the vev $\langle\phi\rangle$. Each Dp-D$(p-4)$ string is now stretched by a different amount, reflecting the fact that each hypermultiplet has a different mass. The potential on the worldvolume theory of the D-instantons is now

$$V = \frac{1}{g^2}\sum_{m,n=5}^{10}[X_m, X_n]^2 + \sum_{m,\mu}[X_m, X_\mu]^2 + \sum_{a=1}^{N}\psi^{a\dagger}(X_m - \phi^m{}_a)^2\psi_a + \tilde{\psi}^a(X_m - \phi^m{}_a)^2\tilde{\psi}^\dagger_a$$
$$+g^2\,\mathrm{Tr}\,(\sum_{a=1}^{N}\psi_a\psi^{a\dagger} - \tilde{\psi}^\dagger_a\tilde{\psi}^a + [Z,Z^\dagger] + [W,W^\dagger])^2 + g^2\,\mathrm{Tr}\,|\sum_{a=1}^{N}\psi_a\tilde{\psi}^a + [Z,W]|^2$$

We've actually introduced more new parameters here than we need, since the D3-branes can be separated in 6 different dimensions, so we have the corresponding positions $\phi^m{}_a$, $m = 4,\ldots,9$ and $a = 1,\ldots N$. Since we have been dealing with just a single scalar field ϕ in this section, we will set $\phi^m_i = 0$ except for $m = 6$ (I know...why 6?!). The parameters $\phi^6{}_a = \phi_a$ are the components of the vev (2.4).

We can now re-examine the vacuum condition for the Higgs branch. If we wish to try to turn on ψ and $\tilde{\psi}$, we must first set $X_m = \phi_a$, for some a. Then the all ψ_b and $\tilde{\psi}^b$ must vanish except for $b = a$. But, taking the trace of the D- and F-term conditions tells us that even ψ_a and $\tilde{\psi}_a$ vanish. We have lost our Higgs branch completely. The interpretation is that the instantons have shrunk to zero size. Note that in the case of non-commutativity, the instantons don't vanish but are pushed to the $U(1)$ instantons with, schematically, $|\psi|^2 \sim \zeta$.

Although the instantons shrink to zero size, there's still important information to be gleaned from the potential above. One can continue to think of the instanton moduli space $\mathcal{I}_{k,N} \cong \mathcal{M}_{\rm Higgs}$ as before, but now with a potential over it. This potential arises after integrating out the X_m and it is not hard to show that it is of a very specific form: it is the length-squared of a triholomorphic Killing vector on $\mathcal{I}_{k,N}$ associated with the $SU(N)$ isometry.

This potential on $\mathcal{I}_{k,N}$ can be derived directly within field theory without recourse to D-branes or the ADHM construction [54]. This is the route we follow here. The question we want to ask is: given an instanton solution, how does the presence of the ϕ vev affect its action? This gives the potential on the instanton moduli space which is simply

$$V = \int d^4x\ \mathrm{Tr}\,(\mathcal{D}_\mu\phi)^2 \tag{2.58}$$

where $\mathcal{D}_\mu$ is evaluated on the background instanton solution. We are allowed to vary ϕ so it minimizes the potential so that, for each solution to the instanton equations, we want to find ϕ such that

$$\mathcal{D}^2\phi = 0 \tag{2.59}$$

with the boundary condition that $\phi \to \langle\phi\rangle$. But we've seen an equation of this form, evaluated on the instanton background, before. When we were discussing the instanton zero modes in section 1.2, we saw that the zero modes arising from the overall $SU(N)$ gauge orientation were of the form $\delta A_\mu = \mathcal{D}_\mu\Lambda$, where Λ tends to a constant at infinity and satisfies the gauge fixing condition $\mathcal{D}_\mu\delta A_\mu = 0$. This means that we can re-write the potential in terms of the overlap of zero modes

$$V = \int d^4x \ \text{Tr}\ \delta A_\mu \delta A_\mu \tag{2.60}$$

for the particular zero mode $\delta A_\mu = \mathcal{D}_\mu\phi$ associated to the gauge orientation of the instanton. We can give a nicer geometrical interpretation to this. Consider the action of the Cartan subalgebra $\vec{H}$ on $\mathcal{I}_{k,N}$ and denote the corresponding Killing vector as $\vec{k} = \vec{k}^\alpha\partial_\alpha$. Then, since ϕ generates the transformation $\vec{\phi}\cdot\vec{H}$, we can express our zero mode in terms of the basis $\delta A_\mu = (\vec{\phi}\cdot\vec{k}^\alpha)\,\delta_\alpha A_\mu$. Putting this in our potential and performing the integral over the zero modes, we have the final expression

$$V = g_{\alpha\beta}\,(\vec{\phi}\cdot\vec{k}^\alpha)\,(\vec{\phi}\cdot\vec{k}^\beta) \tag{2.61}$$

The potential vanishes at the fixed points of the $U(1)^{N-1}$ action. This is the small instanton singularity (or related points on the blown-up cycles in the resolved instanton moduli space). Potentials of the form (2.61) were first discussed by Alvarez-Gaume and Freedman who showed that, for tri-holomorphic Killing vectors k, they are the unique form allowed in a sigma-model preserving eight supercharges [55].

The concept of a potential on the instanton moduli space $\mathcal{I}_{k,N}$ is the modern way of viewing what used to known as the "constrained instanton", that is an approximate instanton-like solution to the theory with $\langle\phi\rangle \neq 0$ [56]. These potentials play an important role in Nekrasov's first-principles computation of the Seiberg-Witten prepotential [59]. Another application occurs in the five-dimensional theory, where instantons are particles. Here the motion on the moduli space may avoid the fate of falling to the zeroes of (2.61) by spinning around the potential like a motorcyclist on the wall of death. These solutions of the low-energy dynamics are dyonic instantons which carry electric charge in five dimensions [54, 57, 58].

2.7 Applications

Time now for the interesting applications, examining the role that monopoles play in the quantum dynamics of supersymmetric gauge theories in various dimensions. We'll look at monopoles in 3, 4, 5 and 6 dimensions in turn.

2.7.1 Monopoles in Three Dimensions

In $d = 2+1$ dimensions, monopoles are finite action solutions to the Euclidean equations of motion and the role they play is the same as that of instantons in $d = 3+1$ dimensions: in a semi-classical evaluation of the path-integral, one must sum over these monopole configurations. In 1975, Polyakov famously showed how a gas of these monopoles leads to linear confinement in non-supersymmetric Georgi-Glashow model [60] (that is, an $SU(2)$ gauge theory broken to $U(1)$ by an adjoint scalar field).

In supersymmetric theories, monopoles give rise to somewhat different physics. The key point is that they now have fermionic zero modes, ensuring that they can only contribute to correlation functions with a suitable number of fermionic insertions to soak up the integrals over the Grassmannian collective coordinates. In $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories[2] in $d = 2 + 1$ dimensions, instantons generate superpotentials, lifting moduli spaces of vacua [61]. In $\mathcal{N} = 8$ theories, instantons contribute to particular 8 fermi correlation functions which have a beautiful interpretation in terms of membrane scattering in M-theory [62, 63]. In this section, I'd like to describe one of the nicest applications of monopoles in three dimensions which occurs in theories with $\mathcal{N} = 4$ supersymmetry, or 8 supercharges.

We'll consider $\mathcal{N} = 4$ $SU(2)$ super Yang-Mills. The superpartners of the gauge field include 3 adjoint scalar fields, ϕ^α, $\alpha = 1, 2, 3$ and 2 adjoint Dirac fermions. When the scalars gain an expectation value $\langle\phi^\alpha\rangle \neq 0$, the gauge group is broken $SU(2) \to U(1)$ and the surviving, massless, bosonic fields are 3 scalars and a photon. However, in $d = 2 + 1$ dimensions, the photon has only a single polarization and can be exchanged in favor of another scalar σ. We achieve this by a duality transformation:

$$F_{ij} = \frac{e^2}{2\pi}\epsilon_{ijk}\partial^k\,\sigma \tag{2.62}$$

[2] A (foot)note on nomenclature. In any dimension, the number of supersymmetries $\mathcal{N}$ counts the number of supersymmetry generators in units of the minimal spinor. In $d = 2+1$ the minimal Majorana spinor has 2 real components. This is in contrast to $d = 3+1$ dimensions where the minimal Majorana (or equivalently Weyl) spinor has 4 real components. This leads to the unfortunate fact that $\mathcal{N} = 1$ in $d = 3+1$ is equivalent to $\mathcal{N} = 2$ in $d = 2+1$. It's annoying. The invariant way to count is in terms of supercharges. Four supercharges means $\mathcal{N} = 1$ in four dimensions or $\mathcal{N} = 2$ in three dimensions.

where we have chosen normalization so that the scalar σ is periodic: $\sigma = \sigma + 2\pi$. Since supersymmetry protects these four scalars against becoming massive, the most general low-energy effective action we can write down is the sigma-model

$$L_{\rm low-energy} = \frac{1}{2e^2}\, g_{\alpha\beta}\, \partial_i \phi^\alpha\, \partial^i \phi^\beta \qquad (2.63)$$

where $\phi^\alpha = (\phi^1, \phi^2, \phi^3, \sigma)$. Remarkably, as shown by Seiberg and Witten, the metric $g_{\alpha\beta}$ can be determined uniquely [27]. It turns out to be an old friend: it is the Atiyah-Hitchin metric (2.30)! The dictionary is $\phi^i = e^2 r^i$ and $\sigma = \psi$. Comparing with the functions a, b and c listed in (2.34), the leading constant term comes from tree level in our 3d gauge theory, and the $1/r$ terms arise from a one-loop correction. Most interesting is the e^{-r} term in (2.34). This comes from a semi-classical monopole computation in $d = 2 + 1$ which can be computed exactly [66]. So we find monopoles arising in two very different ways: firstly as an instanton-like configuration in the 3d theory, and secondly in an auxiliary role as the description of the low-energy dynamics. The underlying reason for this was explained by Hanany and Witten [28], and we shall see a related perspective in section 2.7.4.

So the low-energy dynamics of $\mathcal{N} = 4$ $SU(2)$ gauge theory is dictated by the two monopole moduli space. It can also be shown that the low-energy dynamics of the $\mathcal{N} = 4$ $SU(N)$ gauge theory in $d = 2 + 1$ is governed by a sigma-model on the moduli space of N magnetic monopoles in an $SU(2)$ gauge group [64]. There are 3d quiver gauge theories related to monopoles in higher rank, simply laced (i.e. ADE) gauge groups [28, 65] but, to my knowledge, there is no such correspondence for monopoles in non-simply laced groups.

2.7.2 Monopoles and Duality

Perhaps the most important application of monopoles is the role they play in uncovering the web of dualities relating various theories. Most famous is the S-duality of $\mathcal{N} = 4$ super Yang-Mills in four dimensions. The idea is that we can re write the gauge theory treating magnetic monopoles as elementary particles rather than solitons [67]. The following is a lightening review of this large subject. Many more details can be found in [7].

The conjecture of S-duality states that we may re-express the theory, treating monopoles as the fundamental objects, at the price of inverting the coupling $e \to 4\pi/e$. Since this is a strong-weak coupling duality, we need to have some control over the strong coupling behavior of the theory to test the conjecture. The window on this regime is provided by the BPS states [13], whose mass is not renormalized in the maximally supersymmetric $\mathcal{N}=4$ theory which, among other reasons, makes it a likely place to look for S-duality [68]. In fact, this theory exhibits a more general $SL(2,\mathbf{Z})$ group of duality transformations which acts on the complexified coupling $\tau = \theta/2\pi + 4\pi i/e^2$ by

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad \text{with } a,b,c,d \in \mathbf{Z} \text{ and } ad - bc = 1 \tag{2.64}$$

A transformation of this type mixes up what we mean by electric and magnetic charges. Let's work in the $SU(2)$ gauge theory for simplicity so that electric and magnetic charges in the unbroken $U(1)$ are each specified by an integer (n_e, n_m). Then under the $SL(2,\mathbf{Z})$ transformation,

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \tag{2.65}$$

The conjecture of S-duality has an important prediction that can be tested semi-classically: the spectrum must form multiplets under the $SL(2,Z)$ transformation above. In particular, if S-duality holds, the existence of the W-boson state $(n_e, n_m) = (1,0)$ implies the existence of a slew of further states with quantum numbers $(n_e, n_m) = (a,c)$ where a and c are relatively prime. The states with magnetic charge $n_m = c = 1$ are the dyons that we described in Section 2.3 and can be shown to exist in the quantum spectrum. But we have to work much harder to find the states with magnetic charge $n_m = c > 1$. To do so we must examine the low-energy dynamics of n_m monopoles, described by supersymmetric quantum mechanics on the monopole moduli space. Bound states saturating the Bogomoln'yi bound correspond to ground states of the quantum mechanics. But, as we described in section 1.5.2, this questions translates into the more geometrical search for normalizable harmonic forms on the monopole moduli space.

In the $n_m = 2$ monopole sector, the bound states were explicitly demonstrated to exist by Sen [69]. S-duality predicts the existence of a tower of dyon states with charges $(n_e, 2)$ for all n_e odd which translates into the requirement that there is a unique harmonic form ω on the Atiyah-Hitchin manifold. The electric charge still comes from motion in the $\mathbf{S}^1$ factor of the monopole moduli space (2.29), but the need for only odd charges n_e to exist requires that the form ω is odd under the $\mathbf{Z}_2$ action (2.32). Uniqueness requires that ω is either self-dual or anti-self-dual. In fact, it is the latter. The ansatz,

$$\omega = F(r)(d\sigma_1 - \frac{fa}{bc} dr \wedge \sigma_1) \tag{2.66}$$

is harmonic provided that $F(r)$ satisfies

$$\frac{dF}{dr} = -\frac{fa}{bc}F \tag{2.67}$$

One can show that this form is normalizable, well behaved at the center of the moduli space and, moreover, unique. Historically, the existence of this form was one the compelling pieces of evidence in favor of S-duality, leading ultimately to an understanding of strong coupling behavior of many supersymmetric field theories and string theories.

The discussion above is for $\mathcal{N} = 4$ theories. In $\mathcal{N} = 2$ theories, the bound state described above does not exist (a study of the $\mathcal{N} = (0,4)$ supersymmetric quantum mechanics reveals that the Hilbert space is identified with holomorphic forms and ω is not holomorphic). Nevertheless, there exists a somewhat more subtle duality between electrically and magnetically charged states, captured by the Seiberg-Witten solution [70]. Once again, there is a semi-classical test of these ideas along the lines described above [71].

2.7.3 Monopole Strings and the $(2,0)$ Theory

We've seen that the moduli space of a single monopole is $\mathcal{M} \cong \mathbf{R}^3 \times \mathbf{S}^1$ with metric,

$$ds^2 = M_{\rm mono}\left(dX^i dX^i + \frac{1}{v^2}\,d\chi^2\right) \tag{2.68}$$

where $\chi \in [0, 2\pi)$. It looks as if, at low-energies, the monopole is moving in a higher dimensional space. Is there any situation where we can actually interpret this motion in the $\mathbf{S}^1$ as motion in an extra, hidden dimension of space?

One problem with interpreting internal degrees of freedom, such as χ, in terms of an extra dimension is that there is no guarantee that motion in these directions will be Lorentz covariant. For example, Einstein's speed limit tells us that the motion of the monopole in $\mathbf{R}^3$ is bounded by the speed of light: i.e. $\dot{X} \leq 1$. But is there a similar bound on $\dot{\chi}$? This is an question which goes beyond the moduli space approximation, which keeps only the lowest velocities, but is easily answered since we know the exact spectrum of the dyons. The energy of a relativistically moving dyon is $E^2 = M_{\rm dyon}^2 + p_i p_i$, where p_i is the momentum conjugate to the center of mass X_i. Using the mass formula (2.41), we have the full Hamiltonian

$$H_{\rm dyon} = \sqrt{M_{\rm mono}^2 + v^2 p_\chi^2 + p_i p_i} \tag{2.69}$$

where $p_\chi = 2q$ is the momentum conjugate to χ. This gives rise to the Lagrangian,

$$L_{\rm dyon} = -M_{\rm mono}\sqrt{1 - \dot{\chi}^2/v^2 - \dot{X}^i\dot{X}^i} \tag{2.70}$$

which, at second order in velocities, agrees with the motion on the moduli space (2.68). So, surprisingly, the internal direction χ does appear in a Lorentz covariant manner in this Lagrangian and is therefore a candidate for an extra, hidden, dimension.

However, looking more closely, our hopes are dashed. From (2.70) (or, indeed, from (2.68)), we see that the radius of the extra dimension is proportional to $1/v$. But the width of the monopole core is also $1/v$. This makes it a little hard to convincingly argue that the monopole can happily move in this putative extra dimension since there's no way the dimension can be parametrically larger than the monopole itself. It appears that χ is stuck in the auxiliary role of endowing monopoles with electric charge, rather than being promoted to a physical dimension of space.

Things change somewhat if we consider the monopole as a string-like object in a $d = 4+1$ dimensional gauge theory. Now the low-energy effective action for a single monopole is simply the action (2.70) lifted to the two dimensional worldsheet of the string, yielding the familiar Nambu-Goto action

$$S_{\rm string} = -T_{\rm mono}\int d^2y\ \sqrt{1-(\partial\chi)^2/v^2-(\partial X^i)^2} \tag{2.71}$$

where ∂ denotes derivatives with respect to both worldsheet coordinates, σ and τ. We've rewritten $M_{\rm mono} = T_{\rm mono} = 4\pi v/e^2$ to stress the fact that it is a tension, with dimension 2 (recall that e^2 has dimension -1 in $d = 4+1$). As it stands, we're in no better shape. The size of the circle is still $1/v$, the same as the width of the monopole string. However, now we have a two dimensional worldsheet we may apply T-duality. This means exchanging momentum modes around $\mathbf{S}^1$ for winding modes so that

$$\partial\chi = {}^\star\partial\tilde{\chi} \tag{2.72}$$

We need to be careful with the normalization. A careful study reveals that,

$$\frac{1}{4\pi}\int d^2y\, R^2\,(\partial\chi)^2 \ \rightarrow\ \frac{1}{4\pi}\int d^2y\,\frac{1}{R^2}\,(\partial\tilde{\chi})^2 \tag{2.73}$$

where, up to that important factor of 4π, R is the radius of the circle measured in string units. Comparing with our normalization, we have $R^2 = 8\pi^2/ve^2$, and the dual Lagrangian is

$$S_{\rm string} = -T_{\rm mono}\int d^2y\ \sqrt{1-(e^2/8\pi^2)^2(\partial\tilde{\chi})^2-(\partial X^i)^2} \tag{2.74}$$

We see that the physical radius of this dual circle is now $e^2/8\pi^2$. This can be arbitrarily large and, in particular, much larger than the width of the monopole string. It's a prime candidate to be interpreted as a real, honest, extra dimension. In fact, in the maximally supersymmetric Yang-Mills theory in five dimensions, it is known that this extra dimension is real. It is precisely the hidden circle that takes us up to the six-dimensional $(2,0)$ theory that we discussed in section 1.5.2. The monopole even tells us that the instantons must be the Kaluza-Klein modes since the inverse radius of the dual circle is exactly $M_{\rm inst}$. Once again, we see that solitons allow us to probe important features of the quantum physics where myopic perturbation theory fails. Note that the derivation above does rely on supersymmetry since, for the Hamiltonian (2.69) to be exact, we need the masses of the dyons to saturate the Bogomoln'yi bound (2.41).

2.7.4 D-Branes in Little String Theory

Little string theories are strongly interacting string theories without gravity in $d = 5+1$ dimensions. For a review see [73]. The maximally supersymmetric variety can be thought of as the decoupled theory living on NS5-branes. They come in two flavors: the type iia little string theory is a $(2,0)$ supersymmetric theory which reduces at low-energies to the conformal field theory discussed in sections 1.5.2 and 2.7.3. In contrast, the type iib little string has $(1,1)$ non-chiral supersymmetry and reduces at low-energies to $d = 5+1$ Yang-Mills theory. When this theory sits on the Coulomb branch it admits monopole solutions which, in six dimensions, are membranes. Let's discuss some of the properties of these monopoles in the $SU(2)$ theory.

The low-energy dynamics of a single monopole is the $d = 2+1$ dimensional sigma model with target space $\mathbf{R}^3 \times \mathbf{S}^1$ and metric (2.68). But, as we already discussed, in $d = 2+1$ we can exchange the periodic scalar χ for a $U(1)$ gauge field living on the monopole. Taking care of the normalization, we find

$$F_{mn} = \frac{8\pi^2}{e^2}\epsilon_{mnp}\,\partial^p\chi \tag{2.75}$$

with $m,n = 0,1,2$ denoting the worldvolume dimensions of the monopole 2-brane. The low-energy dynamics of this brane can therefore be written as

$$S_{\rm brane} = \int d^3x\;\frac{1}{2}T_{\rm mono}\left((\partial_m X^i)^2 + \frac{1}{v^2}(\partial_m\chi)^2\right) \tag{2.76}$$

$$= \int d^3x\;\frac{1}{2g^2}\left((\partial_m\varphi^i)^2 + \frac{1}{2}F_{mn}F^{mn}\right) \tag{2.77}$$

where $g^2 = 4\pi^2 T_{\rm mono}/v^2$ is fixed by the duality (2.75) and insisting that the scalar has canonical kinetic term dictates $\varphi^i = (8\pi^2/e^2)\,X^i = T_{\rm inst}X^i$. This normalization will prove important. Including the fermions, we therefore find the low-energy dynamics of a monopole membrane to be free $U(1)$ gauge theory with 8 supercharges (called $\mathcal{N}=4$ in three dimensions), containing a photon and three real scalars.

Six dimensional gauge theories also contain instanton strings. These are the "little strings" of little string theory. We will now show that strings can end on the monopole 2-brane. This is simplest to see from the worldvolume perspective in terms of the original variable χ. Defining the complex coordinate on the membrane worldvolume $z = x^4 + ix^5$, we have the BPS "BIon" spike [74, 75] solution of the theory (2.76)

$$X^1 + \frac{i}{v}\chi = \frac{1}{v}\log(vz) \tag{2.78}$$

Plotting the value of the transverse position X^1 as a function of $|z|$, we see that this solution indeed has the profile of a string ending on the monopole 2-brane. Since χ winds once as we circumvent the origin $z = 0$, after the duality transformation we see that this string sources a radial electric field. In other words, the end of the string is charged under the $U(1)$ gauge field on the brane (2.75). We have found a D-brane in the six-dimensional little string theory.

Having found the string solution from the perspective of the monopole worldvolume theory, we can ask whether we can find a solution in the full $d = 5 + 1$ dimensional theory. In fact, as far as I know, no one has done this. But it is possible to write down the first order equations that this solution must solve [76]. They are the dimensional reduction of equations found in [77] and read

$$\begin{array}{c} F_{23} + F_{45} = \mathcal{D}_1\phi \quad , \quad F_{35} = -F_{42} \quad , \quad F_{34} = -F_{25} \\ F_{31} = \mathcal{D}_2\phi \quad , \quad F_{12} = \mathcal{D}_3\phi \quad , \quad F_{51} = \mathcal{D}_4\phi \quad , \quad F_{14} = \mathcal{D}_5\phi \end{array} \tag{2.79}$$

Notice that among the solutions to these equations are instanton strings stretched in the x^1 directions, and monopole 2-branes with spatial worldvolume (x^4, x^5). It would be interesting to find an explicit solution describing the instanton string ending on the monopole brane.

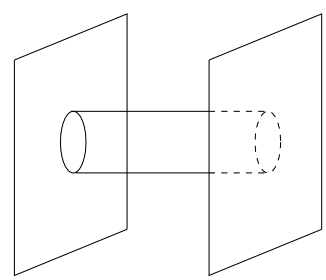

Figure 8:

We find ourselves in a rather familiar situation. We have string-like objects which can terminate on D-brane objects, where their end is electrically charged. Yet all this is within the context of a gauge theory, with no reference to string theory or gravity. Let's remind ourselves about some further properties of D-branes in string theory to see if the analogy can be pushed further. For example, there are two methods to understand the dynamics of D-branes in string theory, using either closed or open strings. The first method — the closed string description — uses the supergravity solution for D-branes to compute their scattering. In contrast, in the second method — the open string description

— the back-reaction on the bulk is ignored. Instead the strings stretched between two branes are integrated in, giving rise to new, light fields of the worldvolume theory as the branes approach. In flat space, this enhances $U(1)^n$ worldvolume gauge symmetry to $U(n)$ [78]. The quantum effects from these non-abelian fields capture the scattering of the D-branes. The equivalence of these two methods is assured by open-closed string duality, where the diagram drawn in figure 8 can be interpreted as tree-level closed string or one-loop open string exchange. Generically the two methods have different regimes of validity.

Is there an analogous treatment for our monopole D-branes? The analogy of the supergravity description is simply the Manton moduli space approximation described in section 2.2. What about the open string description? Can we integrate in the light states arising from instanton strings stretched between two D-branes? They have charge $(+1, -1)$ under the two branes and, by the normalization described above, mass $T_{\rm inst}|X_1^i - X_2^i| = |\varphi_1^i - \varphi_2^i|$. Let's make the simplest assumption that quantization of these strings gives rise to W-bosons, enhancing the worldvolume symmetry of n branes to $U(n)$. Do the quantum effects of these open strings mimic the classical scattering of monopoles? Of course they do! This is precisely the calculation we described in section 2.7.1: the Coulomb branch of the $U(n)$ $\mathcal{N} = 4$ super Yang-Mills in $d = 2 + 1$ dimensions is the n monopole moduli space.

The above discussion is not really new. It is nothing more than the "Hanany-Witten" story [28], with attention focussed on the NS5-brane worldvolume rather than the usual 10-dimensional perspective. Nevertheless, it's interesting that one can formulate the story without reference to 10-dimensional string theory. In particular, if we interpret our results in terms of open-closed string duality summarized in figure 8, it strongly suggests that the bulk six-dimensional Yang-Mills fields can be thought of as quantized loops of instanton strings.

To finish, let me confess that, as one might expect, the closed and open string descriptions have different regimes of validity. The bulk calculation is valid in the full quantum theory only if we can ignore higher derivative corrections to the six-dimensional action. These scale as $e^{2n}\partial^{2n}$. Since the size of the monopole is $\partial \sim v^{-1}$, we have the requirement $v^2e^2 \ll 1$ for the "closed string" description to be valid. What about the open string description? We integrate in an object of energy $E = T_{\rm inst}\Delta X$, where ΔX is the separation between branes. We do not want to include higher excitations of the string which scale as v. So we have $E \ll v$. At the same time, we want $\Delta X > 1/v$, the width of the branes, in order to make sense of the discussion. These two requirements tell us that $v^2e^2 \gg 1$. The reason that the two calculations yield the same result, despite their different regimes of validity, is due to a non-renormalization theorem, which essentially boils down the restrictions imposed by the hyperKähler nature of the metric.

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