3 Grad, Div and Curl

In this section we're going to further develop the ways in which we can differentiate. We'll be particularly interested in how we can differentiate scalar and vector fields. Our definitions will be straightforward but, at least for the time being, we won't be able to offer the full intuition behind these ideas. Perhaps ironically, the full meaning of how to differentiate will become clear only in Section 4 where we also learn the corresponding different ways to integrate.

3.1 The Gradient

We've already seen how to differentiate a scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$. Given Cartesian coordinates x^i with $i = 1, \ldots, n$ on $\mathbb{R}^n$, the gradient of ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\,\mathbf{e}_i \tag{3.1}$$

Note that differentiating a scalar field leaves us with a vector field.

The definition above relies on a choice of Cartesian coordinates. Later in this section, we'll find expressions for the gradient in different coordinate systems. But there is also a definition of the gradient that does not rely on any coordinate choice at all. This starts by considering a point $\mathbf{x} \in \mathbb{R}^n$. We don't, yet, think of $\mathbf{x}$ as defined by a string of n numbers: that comes only with a choice of coordinates. Instead, it should be viewed as an abstract point in $\mathbb{R}^n$.

The first principles, coordinate-free definition of the gradient $\nabla\phi$ simply compares the value of ϕ at some point $\mathbf{x}$ to the value at some neighbouring point $\mathbf{x} + \mathbf{h}$ with $h = |\mathbf{h}| \ll 1$. For a differentiable function ϕ, we can write

$$\phi(\mathbf{x} + \mathbf{h}) = \phi(\mathbf{x}) + \mathbf{h} \cdot \nabla\phi + \mathcal{O}(h^2) \tag{3.2}$$

where this should be thought of as the definition of the gradient $\nabla\phi$. Note that it's similar in spirit to our definition of the tangent to a curve $\dot{\mathbf{x}}$ given in (1.2). If we pick a choice of coordinates, with $\mathbf{x} = (x^1, \ldots, x^n)$, then we can take $\mathbf{h} = \epsilon\,\mathbf{e}_i$ with $\epsilon \ll 1$. The definition (3.2) then coincides with (3.1),

An Example

Consider the function on $\mathbb{R}^3$,

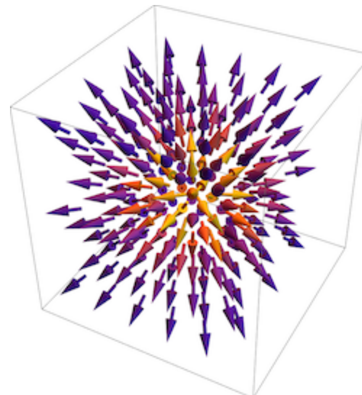
$$\phi(x, y, z) = -\frac{1}{\sqrt{x^2 + y^2 + z^2}} = -\frac{1}{r}$$

where $r^2 = x^2 + y^2 + z^2$ is the distance from the origin. We have

$$\frac{\partial \phi}{\partial x} = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = \frac{x}{r^3}$$

and similar for the others. The gradient is then given by

$$\nabla \phi = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3} = \frac{\hat{\mathbf{r}}}{r^2}$$



where, in the final expression, we've introduced the unit vector $\hat{\mathbf{r}}$ which points out radially outwards in each direction, like the spikes on a hedgehog as shown in the figure. The vector field $\nabla\phi$ points radially, decreasing as $1/r^2$. Vector fields of this kind are important in electromagnetism where they describe the electric field $\mathbf{E}(\mathbf{x})$ arising from a charged particle.

An Application: Following a Curve

Suppose that we're given a curve in $\mathbb{R}^n$, defined by the map $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$, together with a scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$. Then we can combine these into the composite map $\phi(\mathbf{x}(t)) : \mathbb{R} \to \mathbb{R}$. This is simply the value of the scalar field evaluated on the curve. We can then differentiate this map along the curve using the higher dimensional version of the chain rule.

$$\frac{d\phi(\mathbf{x}(t))}{dt} = \frac{\partial \phi}{\partial x^i}\frac{dx^i}{dt}$$

This has a nice, compact expression in terms of the gradient,

$$\frac{d\phi(\mathbf{x}(t))}{dt} = \nabla\phi \cdot \frac{d\mathbf{x}}{dt}$$

This tells us how the function $\phi(\mathbf{x})$ changes as we move along the curve.

3.2 Div and Curl

At this stage we take an interesting and bold mathematical step. We view ∇ as an object in its own right. It is called the *gradient operator*.

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x^i} \tag{3.3}$$

This is both a vector and an operator. The fact that ∇ is an operator means that it's just waiting for a function to come along (from the right) and be differentiated.

The gradient operator ∇ sometimes goes by the names *nabla* or *del*, although usually only when explaining to students in a first course on vector calculus that ∇ sometimes goes by the names *nabla* or *del*. (Admittedly, the latex command for ∇ is \nabla which helps keep the name alive.)

With ∇ divorced from the scalar field on which it originally acted, we can now think creatively about how it may act on other fields. As we've seen, a vector field is defined to be a map

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$$

Given two vectors, we all have a natural urge to dot them together. This gives a derivative acting on vector fields known as the *divergence*

$$\nabla \cdot \mathbf{F} = \left(\mathbf{e}_i \frac{\partial}{\partial x^i} \right) \cdot (\mathbf{e}_j F_j) = \frac{\partial F_i}{\partial x^i}$$

where we've used the orthonormality $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Note that the gradient of a scalar field gave a vector field. Now the divergence of a vector field gives a scalar field.

The divergence isn't the only way to differentiate a vector field. If we're in $\mathbb{R}^n$, a vector field has N components and we could differentiate each of these in one of N different directions. This means that there are N^2 different meanings to the "derivative of a vector field". But the divergence turns out to be the combination that is most useful.

Both the gradient and divergence operations can be applied to a fields in $\mathbb{R}^n$. In contrast, our final operation holds only for vector fields that map

$$\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$$

In this case, we can take the cross product. This gives a derivative of a vector field known as the *curl*,

$$\nabla \times \mathbf{F} = \left(\mathbf{e}_i \frac{\partial}{\partial x^i} \right) \times (\mathbf{e}_j F_j) = \epsilon_{ijk} \frac{\partial F_j}{\partial x^i} \mathbf{e}_k$$

Or, written out in its full glory,

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial x^2} - \frac{\partial F_2}{\partial x^3}, \frac{\partial F_1}{\partial x^3} - \frac{\partial F_3}{\partial x^1}, \frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2} \right) \tag{3.4}$$

The curl of a vector field is, again, a vector field. It can also be written as the determinant

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

As we proceed through these lectures, we'll build intuition for the meaning of these two derivatives. We will see, in particular, that the divergence $\nabla \cdot \mathbf{F}$ measures the net flow of the vector field $\mathbf{F}$ into, or out of, any given point. Meanwhile, the curl $\nabla \times \mathbf{F}$ measures the rotation of the vector field. A full understanding of this will come only in Section 4 when we learn to undo the differentiation through integration. For now we will content ourselves with some simple examples.

Simple Examples

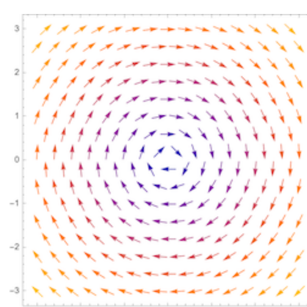
Consider the vector field

$$\mathbf{F}(\mathbf{x}) = (x^2, 0, 0)$$

Clearly this flows in a straight line, with increasing strength. It has $\nabla \cdot \mathbf{F} = 2x$, reflecting the fact that the vector field gets stronger as x increases. It also has $\nabla \times \mathbf{F} = 0$.

Next, consider the vector field

$$\mathbf{F}(\mathbf{x}) = (y, -x, 0)$$



This swirls, as shown in the figure on the right. We have $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = (0, 0, -2)$. The curl points in the $\hat{\mathbf{z}}$ direction, perpendicular to the plane of the swirling.

Finally, we can consider the hedgehog-like radial vector field that we met previously,

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} (x, y, z) \tag{3.5}$$

You can check that this obeys $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$. Or, to be more precise, it obeys these equations *almost* everywhere. Clearly something fishy is going on at the origin $r = 0$. In fact, we will later see that we can make this less fishy: a correct statement is

$$\nabla \cdot \mathbf{F} = 4\pi \delta^3(\mathbf{x})$$

where $\delta^3(\mathbf{x})$ is the higher-dimensional version of the Dirac delta function. We'll understand this result better in Section 5 where we will wield the Gauss divergence theorem.

When evaluating the derivatives of radial fields, like the hedgehog (3.5), it's best to work with the radial distance r, given by $r^2 = x^i x^i$. Taking the derivative then gives $2r\partial r/\partial x^i = 2x^i$ and we have $\partial r/\partial x^i = x^i/r$. You can then check that, for any integer p,

$$\nabla r^p = \mathbf{e}_i \frac{\partial(r^p)}{\partial x^i} = pr^{p-1}\hat{\mathbf{r}}$$

Meanwhile, the vector $\mathbf{x} = x_i \mathbf{e}_i$ can equally well be written as $\mathbf{x} = \mathbf{r} = r\hat{\mathbf{r}}$ which highlights that it points outwards in the radial direction. We have

$$\nabla \cdot \mathbf{r} = \frac{\partial x^i}{\partial x^i} = \delta_{ii} = n$$

where the n arises because we're summing over all $i = 1, \ldots, n$. (Obviously, if we're working in $\mathbb{R}^3$ then $n = 3$.) We can also take the curl

$$\nabla \times \mathbf{r} = \epsilon_{ijk} \frac{\partial x^j}{\partial x^i} \mathbf{e}_k = 0$$

which, of course, as always holds only in $\mathbb{R}^3$.

3.2.1 Some Basic Properties

There are a number of straightforward properties obeyed by grad, div and curl. First, each of these is a *linear* differential operator, meaning that

$$\begin{aligned}
\nabla(\alpha\phi + \psi) &= \alpha\nabla\phi + \nabla\psi \\
\nabla \cdot (\alpha\mathbf{F} + \mathbf{G}) &= \alpha\nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} \\
\nabla \times (\alpha\mathbf{F} + \mathbf{G}) &= \alpha\nabla \times \cdot\mathbf{F} + \nabla \times \mathbf{G}
\end{aligned}$$

for any scalar fields ϕ and ψ, vector fields $\mathbf{F}$ and $\mathbf{G}$, and any constant α.

Next, each of them has a Leibniz properties, which means that they obey a generalisation of the product rule. These are

$$\begin{aligned}
\nabla(\phi\psi) &= \phi\nabla\psi + \psi\nabla\phi \\
\nabla \cdot (\phi\mathbf{F}) &= (\nabla\phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F}) \\
\nabla \times (\phi\mathbf{F}) &= (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})
\end{aligned}$$

In the last of these, you need to be careful about the placing and ordering of ∇, just like you need to be careful about the ordering of any other vector when dealing with the cross product. The proof of any of these is simply an exercise in plugging in the component definition of the operator and using the product rule. For example, we can prove the second equality thus:

$$\nabla \cdot (\phi \mathbf{F}) = \frac{\partial(\phi F_i)}{\partial x^i} = \frac{\partial \phi}{\partial x^i} F_i + \phi \frac{\partial F_i}{\partial x^i} = (\nabla \phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$$

There are also a handful of further Leibnizian properties involving two vector fields. The first of these is straightforward to state:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

This is simplest to prove using index notation. Alternatively, it follows from the usual scalar triple product formula for three vectors. To state the other properties, we need one further small abstraction. Given a vector field $\mathbf{F}$ and the gradient operator ∇, we can construct further differential operators. These are

$$\mathbf{F} \cdot \nabla = F_i \frac{\partial}{\partial x^i} \quad \text{and} \quad \mathbf{F} \times \nabla = \mathbf{e}_k \epsilon_{ijk} F_i \frac{\partial}{\partial x^j}$$

Note that the vector field $\mathbf{F}$ sits on the left, so isn't acted upon by the partial derivative. Instead, each of these objects is itself a differential operator, just waiting for something to come along so that it can differentiate it. In particular, these constructions appear in two further identities

$$\begin{aligned} \nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} \\ \nabla \times (\mathbf{F} \times \mathbf{G}) &= (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \end{aligned}$$

Again, these are not difficult to prove: they follow from expanding out the left-hand side in components.

3.2.2 Conservative is Irrotational

Recall that a conservative vector field $\mathbf{F}$ is one that can be written as

$$\mathbf{F} = \nabla \phi$$

for some scalar field ϕ. We also say that $\mathbf{F}$ is *irrotational* if $\nabla \times \mathbf{F} = 0$. There is a beautiful theorem that says these two concepts are actually equivalent:

Theorem: (Poincaré lemma) For fields defined *everywhere* on $\mathbb{R}^3$, conservative is the same as irrotational.

$$\nabla \times \mathbf{F} = 0 \quad \Longleftrightarrow \quad \mathbf{F} = \nabla \phi$$

Half Proof: It is trivial to prove this in one direction, Suppose that $\mathbf{F} = \nabla\phi$, so that $F_i = \partial_i\phi$. Then

$$\nabla \times \mathbf{F} = \epsilon_{ijk}\partial_i F_j \mathbf{e}_k = \epsilon_{ijk}\partial_i\partial_j\phi\, \mathbf{e}_k = 0$$

which vanishes because the ϵ_{ijk} symbol means that we're anti-symmetrising over ij, but the partial derivatives $\partial_i\partial_j$ are symmetric, so the terms like $\partial_1\partial_2 - \partial_2\partial_1$ cancel.

It is less obvious that the converse statement holds, i.e. that irrotational implies conservative. We'll show this only in Section 4.4 where it appears as a corollary of Stokes' theorem. □

Recall that in Section 1.3 we showed that the line integral of a conservative field was independent of the path taken. Putting this together with the result above, we have we have the following, equivalent statements:

$$\nabla \times \mathbf{F} = 0 \quad \Longleftrightarrow \quad \mathbf{F} = \nabla\phi \quad \Longleftrightarrow \quad \oint_C \mathbf{F}\cdot d\mathbf{x} = 0$$

where we've yet to see the proof of the first $\Longrightarrow$.

3.2.3 Solenoidal Fields

Here is another definition. A vector field $\mathbf{F}$ is called *divergence free* or *solenoidal* if $\nabla\cdot\mathbf{F} = 0$. (The latter name comes from electromagnetism, where a magnetic field $\mathbf{B}$ is most easily generated by a tube with a bunch of wires wrapped around it known as a "solenoid" and has the property $\nabla\cdot\mathbf{B} = 0$.)

There is a nice theorem about divergence free fields that is a counterpart to the one above:

Theorem: Any divergence free field can be written as the curl of something else,

$$\nabla\cdot\mathbf{F} = 0 \quad \Longleftrightarrow \quad \mathbf{F} = \nabla\times\mathbf{A}$$

again, provided that $\mathbf{F}$ is defined everywhere on $\mathbb{R}^3$. Note that $\mathbf{A}$ is not unique. In particular, if you find one $\mathbf{A}$ that does the job then any other $\mathbf{A}+\nabla\phi$ will work equally as well.

Proof: It's again straightforward to show this one way. If $\mathbf{F} = \nabla\times\mathbf{A}$, then $F_i = \epsilon_{ijk}\partial_j A_k$ and so

$$\nabla\cdot\mathbf{F} = \partial_i(\epsilon_{ijk}\partial_j A_k) = 0$$

which again vanishes for the symmetry reasons.

This time, we will prove the converse statement by explicitly exhibiting a vector potential $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$. We pick some arbitrary point $\mathbf{x}_0 = (x_0, y_0, z_0)$ and then construct the following vector field

$$\mathbf{A}(\mathbf{x}) = \left(\int_{z_0}^{z} F_y(x, y, z')\, dz' \;,\; \int_{x_0}^{x} F_z(x', y, z_0)\, dx' - \int_{z_0}^{z} F_x(x, y, z')\, dz' \;,\; 0 \right) \tag{3.6}$$

Since $A_z = 0$, the definition of the curl (3.4) becomes

$$\nabla \times \mathbf{A} = \left(-\frac{\partial A_y}{\partial z} \;,\; \frac{\partial A_x}{\partial z} \;,\; \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Using the ansatz (3.6), we find that the first two components of $\nabla \times \mathbf{A}$ immediately give what we want

$$(\nabla \times \mathbf{A})_x = F_x(x, y, z) \quad \text{and} \quad (\nabla \times \mathbf{A})_y = F_y(x, y, z)$$

both of which follow from the fundamental theorem of calculus. Meanwhile, we still have a little work ahead of us for the final component

$$(\nabla \times \mathbf{A})_z = F_z(x, y, z_0) - \int_{z_0}^{z} \frac{\partial F_x}{\partial x}(x, y, z')\, dz' - \int_{z_0}^{z} \frac{\partial F_y}{\partial y}(x, y, z')\, dz'$$

At this point we use the fact that $\mathbf{F}$ is solenoidal, so $\nabla \cdot \mathbf{F} = 0$ and so $\partial F_z / \partial z = -(\partial F_x / \partial x + \partial F_y / \partial y)$. We then have

$$(\nabla \times \mathbf{A})_z = F_z(x, y, z_0) + \int_{z_0}^{z} \frac{\partial F_z}{\partial z}(x, y, z')\, dz' = F_z(x, y, z)$$

This is the result we want. □

Note that both theorems above come with a caveat: the fields must be defined everywhere on $\mathbb{R}^3$. This is important as counterexamples exist that do not satisfy this requirement, similar to the one that we met in a previous context in Section 1.3.4. These counterexamples will take on a life of their own in future courses where they provide the foundations to think about topology, both in mathematics and physics.

3.2.4 The Laplacian

The *Laplacian* is a second order differential operator defined by

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^i \partial x^i}$$

For example, in 3d the Laplacian takes the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This is a scalar differential operator meaning that, when acting on a scalar field ϕ, it gives back another scalar field $\nabla^2\phi$. Similarly, it acts component by component on a vector field $\mathbf{F}$, giving back another vector field $\nabla^2\mathbf{F}$. If we use the vector triple product formula, we find

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$

which we can rearrange to give an alternative expression for the Laplacian acting on the components of a vector field

$$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$$

We'll devote Section 5 to solving various equations involving the Laplacian.

3.2.5 Some Vector Calculus Equations in Physics

I mentioned in the introduction that all laws of physics are written in the language of vector calculus (or, in the case of general relativity, a version of vector calculus extended to curved spaces, known as differential geometry). Here, for example, are the four equations of electromagnetism, known collectively as the *Maxwell equations*

$$\begin{aligned} \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad &, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \quad &, \quad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \end{aligned} \tag{3.7}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, while $\rho(\mathbf{x})$ is a scalar field that describes the distribution of electric charge in space and $\mathbf{J}(\mathbf{x})$ is a vector field that describes the distribution of electric currents. The equations also include two constants of nature, ϵ_0 and μ_0 which describe the strengths of the electric and magnetic forces respectively.

This simple set of equations describes everything we know about the electricity, magnetism and light. Extracting this information requires the tools that we will develop in the rest of these lectures. Along the way, we will sometimes turn to the Maxwell equations to illustrate new ideas.

You'll find the Laplacian sitting in many other equations of physics. For example, the Schrödinger equation describing a quantum particle is written using the Laplacian. A particularly important equation, that crops up in many places, is the *heat equation*,

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

This tells us, for example, how temperature $T(\mathbf{x}, t)$ evolves over time. Here D is called the *diffusion constant*. This same equation also governs the spread of many other substances when there is some random element in the process, such as the constant bombardment from other atoms. For example, the smell of that guy who didn't shower before coming to lectures spreads through the room in manner described by the heat equation.

3.3 Orthogonal Curvilinear Coordinates

The definition of all our differential operators relied heavily on using Cartesian coordinates. The purpose of this section is simply to ask what these objects look like in different coordinate systems. As usual, the spherical polar and cylindrical polar coordinates in $\mathbb{R}^3$ will be of particular interest to us.

In general, we can describe a point $\mathbf{x}$ in $\mathbb{R}^3$ using some coordinates u, v, w, so $\mathbf{x} = \mathbf{x}(u, v, w)$. Changing either of these coordinates, leaving the others fixed, results in a change in $\mathbf{x}$. We have

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u}\, du + \frac{\partial \mathbf{x}}{\partial v}\, dv + \frac{\partial \mathbf{x}}{\partial w}\, dw \tag{3.8}$$

Here $\partial\mathbf{x}/\partial u$ is the tangent vector to the lines defined by $v, w =$ constant, with similar statements for the others. A given set of coordinates provides a good parameterisation of some region provided that

$$\frac{\partial \mathbf{x}}{\partial u} \cdot \left(\frac{\partial \mathbf{x}}{\partial v} \times \frac{\partial \mathbf{x}}{\partial w} \right) \neq 0$$

The coordinate (u, v, w) are said to be *orthogonal curvilinear* if the three tangent vectors are mutually orthogonal. Here the slightly odd name "curvilinear" reflects the fact that these tangent vectors are typically not constant, but instead depend on position. We'll see examples shortly.

For orthogonal curvilinear coordinates, we can always define orthonormal tangent vectors simply by normalising them. We write

$$\frac{\partial \mathbf{x}}{\partial u} = h_u \mathbf{e}_u \quad , \quad \frac{\partial \mathbf{x}}{\partial v} = h_v \mathbf{e}_v \quad , \quad \frac{\partial \mathbf{x}}{\partial w} = h_w \mathbf{e}_w$$

where we've introduced *scale factors* $h_u, h_v, h_w > 0$ and $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$ form a right-handed orthonormal basis so that $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$. This can always be achieved simply by ordering the coordinates appropriately. Our original equation (3.8) can now be written as

$$d\mathbf{x} = h_u \mathbf{e}_u du + h_v \mathbf{e}_v dv + h_w \mathbf{e}_w dw \tag{3.9}$$

Squaring this, we have

$$d\mathbf{x}^2 = h_u^2\, du^2 + h_v^2\, dv^2 + h_w^2\, dw^2$$

from which it's clear that h_u, h_v and h_w are scale factors that tell us the change in length as we change each of the coordinates.

Throughout this section, we'll illustrate everything with three coordinate systems.

Cartesian Coordinates

First, Cartesian coordinates are easy:

$$\mathbf{x} = (x, y, z) \quad \Longrightarrow \quad h_x = h_y = h_z = 1 \quad \text{and} \quad \mathbf{e}_x = \hat{\mathbf{x}},\ \mathbf{e}_y = \hat{\mathbf{y}},\ \mathbf{e}_z = \hat{\mathbf{z}}$$

Cylindrical Polar Coordinates

Next, cylindrical polar coordinates are defined by (see also (2.7))

$$\mathbf{x} = (\rho \cos\phi, \rho \sin\phi, z)$$

with $\rho \geq 0$ and $\phi \in [0, 2\pi)$ and $z \in \mathbb{R}$. Inverting,

$$\rho = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\phi = \frac{y}{x}$$

It's straightforward to calculate

$$\begin{aligned} \mathbf{e}_\rho &= \hat{\boldsymbol{\rho}} = (\cos\phi, \sin\phi, 0) \\ \mathbf{e}_\phi &= \hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi, 0) \\ \mathbf{e}_z &= \hat{\mathbf{z}} \end{aligned}$$

with

$$h_\rho = h_z = 1 \quad \text{and} \quad h_\phi = \rho$$

The three orthonormal vectors are shown on the left-hand side of Figure 12 in red. Note, in particular, that the vectors depend on ϕ and rotate as you change the point at which they're evaluated.

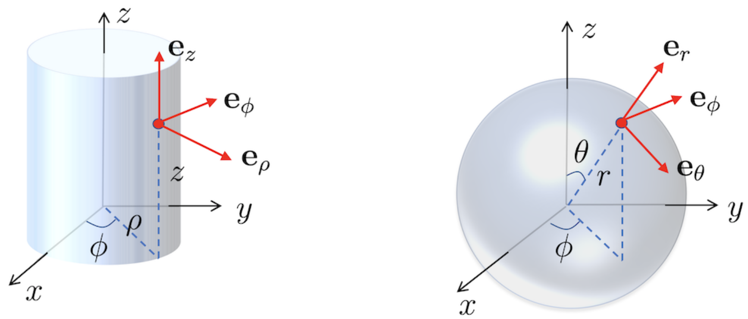


Figure 12. Cylindrical polar coordinates, on the left, and spherical polar coordinates, on the right.

Spherical Polar Coordinates

Spherical polar coordinates are defined by (see also (2.5).)

$$\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

with $r \geq 0$ and $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Inverting,

$$r = \sqrt{x^2 + y^2 + z^2} \quad , \quad \tan\theta = \frac{\sqrt{x^2+y^2}}{z} \quad , \quad \tan\phi = \frac{y}{x}$$

Again, we can easily calculate the basis vectors

$$\begin{aligned} \mathbf{e}_r &= \hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \mathbf{e}_\theta &= \hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \mathbf{e}_\phi &= \hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi, 0) \end{aligned}$$

These are shown in the right-hand side of Figure 12 in red. This time, the scaling factors are

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r\sin\theta$$

We'll now see how various vector operators appear when written in polar coordinates.

3.3.1 Grad

The gradient operator is straightforward. If we shift the position from $\mathbf{x}$ to $\mathbf{x} + \delta\mathbf{x}$, then a scalar field $f(\mathbf{x})$ changes by

$$df = \nabla f \cdot d\mathbf{x} \tag{3.10}$$

This definition can now be used in any coordinate system. In a general coordinate system we have

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw = \nabla f \cdot (h_u \mathbf{e}_u du + h_v \mathbf{e}_v dv + h_w \mathbf{e}_w dw)$$

Using the orthonormality of the basis elements vectors, and comparing the terms on the left and right, this then gives us the gradient operator

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w \tag{3.11}$$

In cylindrical polar coordinates, the gradient of a function $f(\rho, \phi, z)$ is

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

In spherical polar coordinates, the gradient of a function $f(r, \theta, \phi)$ is

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Note, in particular, that when we differentiate with respect to an angle there is always a compensating 1/length prefactor to make sure that the dimensions are right.

3.3.2 Div and Curl

To construct the div and curl in a general coordinate system, we first extract the vector differential operator

$$\nabla = \frac{1}{h_u}\mathbf{e}_u\frac{\partial}{\partial u} + \frac{1}{h_v}\mathbf{e}_v\frac{\partial}{\partial v} + \frac{1}{h_w}\mathbf{e}_w\frac{\partial}{\partial w} \tag{3.12}$$

where, importantly, we've placed the vectors to the left of the differentials because, as we've seen, the basic vectors now typically depend on the coordinates. If we act on a function f with this operator, we recover the gradient (3.11). But now we have this abstract operator, we can also take it to act on a vector field $\mathbf{F}(u, v, w)$. We can expand the vector field as

$$\mathbf{F}(u, v, w) = F_u\mathbf{e}_u + F_v\mathbf{e}_v + F_w\mathbf{e}_w$$

Each of the components depends on the coordinates u, v and w. But so too, in general, do the basis vectors $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$. This means that when the derivatives in the differential operator (3.12) hit $\mathbf{F}$, they also act on both the components and the basis vectors.

Given an explicit expression for the basis vectors, it's not hard to see what happens when they are differentiated. For example, in cylindrical polar coordinates we find

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

and

$$\nabla \times \mathbf{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$$

There is a question on Examples Sheet 2 that asks you to explicitly verify this. Meanwhile, in spherical polar coordinates, we have

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

and

$$\begin{aligned}\nabla \times \mathbf{F} = &\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta F_\phi)}{\partial\theta} - \frac{\partial F_\theta}{\partial \phi}\right)\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial(rF_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}}\end{aligned}$$

For completeness, we also give the general results

Claim: Given a vector field $\mathbf{F}(u,v,w)$ in a general orthogonal, curvilinear coordinate system, the divergence is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right) \tag{3.13}$$

and the curl is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{e}_u & h_v\mathbf{e}_v & h_w\mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

where the derivatives on the second line should now be thought of as acting on the third line only, but not the first. This means that, in components, we have

$$\nabla \times \mathbf{F} = \frac{1}{h_v h_w}\left(\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v)\right) + \text{two similar terms}$$

Proof: Not now. Later. It turns out to be a little easier when we have some integral technology in hand. For this reason, we'll revisit this in Section 4.4.4.

3.3.3 The Laplacian

Finally, we have the Laplacian. From (3.11) and (3.13), this takes the general form

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

Obviously in Cartesian coordinates, the Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In cylindrical polar coordinates it takes the form

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{3.14}$$

and in spherical polar coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{3.15}$$

The most canonical of canonical physics textbooks is J.D. Jackson's "Classical Electrodynamics". I don't know of any theoretical physicist who doesn't have a copy on their shelf. It's an impressive book but I'm pretty sure that, for many, the main selling point is that it has these expressions for div, grad and curl in cylindrical and polar coordinates printed on the inside cover. We'll return to the Laplacian in different coordinate systems in Section 5.2 where we'll explore the solutions to equations like $\nabla^2 f = 0$.