Vector Calculus: Example Sheet 2

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The material necessary to attempt all questions will have been covered by the end of lecture 13.

1. Obtain the equation of the plane which is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

2. The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals of a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that $\mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0$.

The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $\kappa = |\mathbf{H} \times (\mathbf{H} \cdot \nabla)\mathbf{H}|/|\mathbf{H}^3|$.

3. Let $\phi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (\phi \mathbf{v}) = (\nabla \phi) \cdot \mathbf{v} + \phi(\nabla \cdot \mathbf{v}), \quad \nabla \times (\phi \mathbf{v}) = (\nabla \phi) \times \mathbf{v} + \phi(\nabla \times \mathbf{v}).$$

Evaluate the divergence and curl of the following:

$$r\mathbf{x}, \quad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \quad \mathbf{a} \times \mathbf{x}, \quad \mathbf{x}/r^3,$$

where $r = |\mathbf{x}|$ and $\mathbf{a}, \mathbf{b}$ are constant vectors.

4. For vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$, use suffix notation to show that,

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{v},$$

Show also that

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

5. Verify directly that the vector field

$$\mathbf{u}(\mathbf{x}) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)$$

is *irrotational* and express is as the gradient of a scalar field ϕ. Check that $\mathbf{u}$ is *solenoidal* and show that it can be written as the curl of the vector field $\mathbf{v} = (0, 0, \psi)$, for some function ψ.

6. Check that the following vector field is irrotational

$$\mathbf{F} = (3x^2 \tan z - y^2 e^{-xy^2} \sin y \ , \ (\cos y - 2xy \sin y) e^{-xy^2} \ , \ x^3 \sec^2 z)$$

Find the most general scalar potential $\phi(\mathbf{x})$ such that $\mathbf{F} = \nabla \phi$.

7*. Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is divergence free, i.e. $\nabla \cdot \mathbf{F} = 0$. Show that $\mathbf{F} = \nabla \times \mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})\, \mathrm{d}t.$$

What goes wrong with this formula if $\mathbf{F}$ is not defined on all of $\mathbb{R}^3$?

8. Let (u, v, w) be a set of orthogonal curvilinear coordinates for $\mathbb{R}^3$. Show that

$$\mathrm{d}V = h_u h_v h_w \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w.$$

Confirm that $\mathrm{d}V = \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}z$ and $\mathrm{d}V = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$ an cylindrical and spherical polars respectively.

9. If $\mathbf{a}$ is constant vector and $r = |\mathbf{x}|$, verify that

$$\nabla (r^n) = n r^{n-2} \mathbf{x}, \qquad \nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$$

using (i) Cartesian coordinates and suffix notation, (ii) cylindrical polar coordinates, (iii) spherical polar coordinates, and (iv) a first principles, coordinate-independent approach.

[Note: for parts (ii) and (iii) you will need to be careful with the components of $\mathbf{a}$ with respect to each of the relevant bases.]

10. The vector field $\mathbf{A}(\mathbf{x})$ is, in Cartesian, cylindrical and spherical polar coordinates respectively,

$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2} y \, \mathbf{e}_x + \frac{1}{2} x \, \mathbf{e}_y = \frac{1}{2} \rho \, \mathbf{e}_\phi = \frac{1}{2} r \sin\theta \, \mathbf{e}_\phi.$$

Compute the $\nabla \times \mathbf{A}$ in each different coordinate system and check that your answers agree.

11. Recall that in cylindrical polar coordinates

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho,$$

while all other derivatives are zero. Derive expressions for the $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ where $\mathbf{A}$ is an arbitrary vector field given in cylindrical polars by $\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z$. Also derive an expression for the Laplacian of a scalar function $\nabla^2 f$ in this coordinate system

12. By applying the divergence theorem to the vector field $\mathbf{a} \times \mathbf{A}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \nabla \times \mathbf{A} \, \mathrm{d}V = \int_S \mathrm{d}\mathbf{S} \times \mathbf{A}$$

where $S = \partial V$. Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $\mathbf{A}(\mathbf{x}) = (z, 0, 0)$.

13. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let S be the *open* surface

$$1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.$$

Use the divergence theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. Verify your result by calculating the area integral directly.

[Hint: you should find that $\mathrm{d}\mathbf{S} = (2\rho\cos\phi, 2\rho\sin\phi, 1)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$.]

14. For the electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ define the quantities

$$U = \frac{1}{2}\left(\varepsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right), \quad \mathbf{P} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

Use Maxwell's equations with $\mathbf{J} = 0$ to establish the conservation law $\partial U/\partial t + \nabla \cdot \mathbf{P} = 0$. If $U(\mathbf{x})$ has the interpretation of the energy density stored in electric and magnetic fields, what is the interpretation of the so-called Poynting vector $\mathbf{P}$?